

HOMOTOPY THEORY FOR ALGEBRAS OVER POLYNOMIAL MONADS

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ABSTRACT. We study the existence and left properness of transferred model structures for “monoid-like” objects in monoidal model categories. These include genuine monoids, but also all kinds of operads, e.g. symmetric, cyclic, modular, n -operads, properads and PROP’s. All these structures can be realised as algebras over polynomial monads. To get a useful criterion for left properness we introduce a model-theoretical concept of h -cofibration and call a model category h -monoidal if it is monoidal and if the tensor of a (trivial) cofibration with an object yields a (trivial) h -cofibration. Most of the model categories used in algebraic topology or higher category theory are h -monoidal.

We give a general condition for a polynomial monad which ensures the existence and (relative) left properness of a transferred model structure for its algebras. This condition is of a combinatorial nature and singles out a special class of polynomial monads which we call tame polynomial. Many important monads are shown to be tame polynomial. On the other hand there are interesting polynomial monads which are not tame, as for example the monads for modular operads, properads or PROPs. We show that the failure of tameness produces obstructions for the existence of a transfer.

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INTRODUCTION

This text emerged from model-theoretical properties needed in our approach to the Stabilisation Hypothesis of Baez-Dolan [1]. Indeed, our proof [8] of the Stabilisation Hypothesis relies on the existence of a certain left Bousfield localisation of the transferred model structure on n -operads [4], which turns n -operads into higher categorical analogs of E_n -operads ([6, 15]). However, the available techniques for left Bousfield localisation ([29]) require left properness, and it turned out to be surprisingly difficult to verify this property for n -operads. In this article we address among others the general problem of *preservation of left properness under transfer*, and show that, under some specific conditions on the base category, a certain form of preservation is given for an interesting class of transfers, including those for the known model structures on monoids [46], reduced symmetric operads [10], general non-symmetric operads [41], reduced n -operads [4].

The common feature of all these transfers is that in each case the algebraic structure is governed by a *polynomial monad* in sets. Building on 2-categorical techniques developed in [5, 9], we show here that for this kind of algebraic structure, the *existence of a transferred model structure* and its *left properness* are intimately related. Both rely on a careful analysis of free algebra extensions. In the case of free monoid extensions this analysis has been done by Schwede-Shipley [46] in an exemplary and prototypical way. It is among the main results of this article that an analogous analysis is available for free algebra extensions over a general polynomial monad, provided the latter satisfies an extra-condition. This condition is of a combinatorial nature: it requires a certain category (attached to the polynomial monad) to be a coproduct of categories with terminal object. A polynomial monad fulfilling this extra-condition will be called *tame*.

All operads above are algebras over tame polynomial monads. In particular, we recover Muro's [41] recent construction of free non-symmetric operad extensions. Although non-symmetric operads can be viewed as monoids for a certain circle-product, the construction of these free operad extensions is highly non-trivial, since the circle-product commutes with colimits only on one side, while the Schwede-Shipley construction of free monoid extensions is based on a commutation with colimits on both sides. The availability of a Schwede-Shipley type construction for free algebra extensions depends on the behaviour of what we call *semi-free coproducts*. These are coproducts of an algebra with a free algebra. Any *tame* polynomial monad induces a "polynomial expansion" for semi-free coproducts. E. g., the underlying object of the coproduct $M \vee F_T(K)$ of a monoid M with a free

monoid $F_T(K)$ can be computed as follows (cf. [46] and Section 10):

$$(1) \quad M \vee F_T(K) = \coprod_{n \geq 0} M \otimes (K \otimes M)^{\otimes n}.$$

The existence of an analogous functorial “polynomial expansion” for semi-free coproducts of T -algebras over a tame polynomial monad T is the main ingredient for a transferred model structure on T -algebras with good properties.

Interestingly, the question of left properness has not been dealt with in literature, even in the basic case of monoids. Although the *monoid axiom* of Schwede-Shipley [46] gives a quite precise criterion for the existence of a transfer, the monoid axiom alone does not guarantee preservation of left properness under transfer. We propose in this article a common strengthening of the monoid axiom and of left properness which ensures that the transferred model structure on T -algebras is left proper. We also show that a relative form of left properness (namely, weak equivalences between T -algebras with *cofibrant underlying object* are closed under pushout along cofibrations) is given under more general conditions.

This strengthening crucially involves a model-theoretical concept of *h-cofibration*. We call a model category *h-monoidal* if it is a monoidal model category [30] and the tensor product of a (trivial) cofibration with an arbitrary object is a (trivial) *h-cofibration*. An *h-monoidal* model category, which is *compactly generated* [12], satisfies the monoid axiom of Schwede-Shipley. Most of the model categories of algebraic topologists are compactly generated *h-monoidal*. If in addition the class of weak equivalence is closed under tensor product (e.g. all objects are cofibrant) then the model category is called *strongly h-monoidal*.

Our main theorem can now be stated as follows:

Theorem 0.1. *For any tame polynomial monad T in sets and any compactly generated monoidal model category \mathcal{E} fulfilling the monoid axiom, the category of T -algebras in \mathcal{E} admits a relatively left proper transferred model structure.*

The transferred model structure is left proper provided \mathcal{E} is strongly h-monoidal.

Examples of T -algebras in \mathcal{E} for tame polynomial monads T include monoids, non-symmetric operads, reduced symmetric operads, reduced n -operads, reduced cyclic operads, as well as higher opetopic extensions of all these structures. In several of these cases, existence results for a transferred model structure (under some conditions on \mathcal{E}) were known before. It seems however that even in the known cases, our assumptions on \mathcal{E} are weaker than those which appeared in literature. Moreover, the discussion of left properness seems to be completely new. Even more importantly, the uniformity of our approach allows us to give explicit formulas for the *total left derived functors* induced by morphisms of polynomial monads. These explicit formulas have often concrete applications.

Failure of tameness for a polynomial monad very often produces obstructions for the existence of transfer. However, these obstructions can be removed by imposing more restrictive conditions on the base category \mathcal{E} . For instance, if \mathcal{E} is the category of chain complexes over a field of characteristic 0 then a transferred model structure on modular operads, properads or wheeled PROPs in \mathcal{E} exists, even though the corresponding polynomial monads are not tame.

The article is subdivided into three rather independent parts.

Part 1 develops basic properties of h -monoidal model categories and relates this notion to the monoid axiom of Schwede-Shipley [46]. We recall the concept of compact generation [12] of a monoidal model category and give general “admissibility” conditions on a monad, sufficient for the existence and relative left properness of the transfer. Two themes are treated in some detail: monoids in h -monoidal model categories (closely following Schwede-Shipley [46] but adding left properness) and stability of h -monoidality under passage to “convoluted” diagram categories (here we extend some of the results of Dundas-Oestvar-Roendigs [18]).

Part 2 is devoted to polynomial monads. This part relies on 2-categorical techniques developed in [5, 9], but we have tried to keep the presentation as self-contained as possible. These techniques are used to reformulate the construction of a pushout along a free T -algebra map as a left Kan extension of a certain functor attached to T . More generally, given a morphism of polynomial monads $S \rightarrow T$, the induced functor $\text{Alg}_S(\mathcal{E}) \rightarrow \text{Alg}_T(\mathcal{E})$, left adjoint to restriction, is expressible as a left Kan extension. Our main theorem then follows by combining this construction with the results of Part 1, since the explicit formula for free algebra extensions over a tame polynomial monad implies the admissibility of the latter.

At the end we study the Quillen adjunction induced by a morphism of tame polynomial monads. We show that in good cases the total left derived functor can be calculated as a homotopy colimit. Instances of this appear in [4], where a higher-categorical Eckmann-Hilton argument is used to show that the derived symmetrization of the terminal n -operad is homotopy-equivalent to the Fulton-MacPherson compactification of point configurations in \mathbb{R}^n ; and in [25], where Gian-siracusa computes the derived modular envelope of several cyclic operads. Doing so he closely follows Costello [16] who suggested that the derived modular envelope of the terminal planar cyclic operad is homotopy equivalent to the modular operad of nodal Riemann spheres with boundary. Notice that the main obstacle for Gian-siracusa and Costello in rendering such a statement precise was the missing model structure on cyclic operads. This is by now not anymore the case.

Part 3 studies examples. We first show that the monads for monoids and for non-symmetric operads are tame. We then discuss the monad for reduced n -operads. In [5, 4], the latter was shown to be polynomial. Here we show that it is tame, which is quite a bit harder. This particular example is of special interest for us for reasons explained at the beginning of this introduction. In fact, it was this example which motivated the whole project. We also consider other examples of tame polynomial monads, namely those for reduced symmetric operads and reduced cyclic operads. In all these cases, there is a transferred model structure with good properties. We also show that the polynomial monads for general (non-reduced) symmetric, cyclic and n -operads for $n \geq 2$, are not tame. We finally show that the polynomial monads for operads based on graphs rather than trees (such as modular operads, properads or PROPs) are not tame, even if we consider reduced or normalized versions. For all these operad types there is no transferred model structure for chain operads in positive characteristics. Nevertheless, a transfer often exists in characteristic 0.

Part 1. Model structure for algebras over admissible monads

1. HOMOTOPY COFIBRATIONS AND h -MONOIDAL MODEL CATEGORIES

We introduce and investigate here a model-theoretical concept of h -cofibration which seems interesting in itself. The dual concept of an h -fibration has been studied by Rezk [43] under the name of sharp map. The definition of an h -cofibration only depends on the class of weak equivalences and on the existence of pushouts. In left proper model categories, the class of h -cofibrations can be considered as the closure of the class of cofibrations under *cofiber equivalence*, cf. Proposition 1.5 below. We will mainly use h -cofibrations in order to formulate a strengthening of left properness, well adapted to *monoidal* model categories. A similar concept has been developed by Dundas-Ostvaer-Roendigs [18, Definition 4.6].

For an object X of a category \mathcal{E} the undercategory X/\mathcal{E} has as objects morphisms with domain X , and as morphisms “commuting triangles” under X . If \mathcal{E} carries a model structure then so does X/\mathcal{E} . A map $X \rightarrow A \rightarrow B$ in X/\mathcal{E} is a cofibration, weak equivalence, resp. fibration if and only if the underlying map $A \rightarrow B$ in \mathcal{E} is.

Definition 1.1. *A morphism $f : X \rightarrow Y$ in a model category \mathcal{E} is called an h -cofibration if the functor $f_! : X/\mathcal{E} \rightarrow Y/\mathcal{E}$ (given by cobase change along f) preserves weak equivalences.*

In more explicit terms, a morphism $f : X \rightarrow Y$ in \mathcal{E} is an h -cofibration if and only if in each commuting diagram of pushout squares in \mathcal{E}

$$(2) \quad \begin{array}{ccccc} X & \longrightarrow & A & \xrightarrow{w} & B \\ f \downarrow & & \downarrow & & \downarrow \\ Y & \longrightarrow & A' & \xrightarrow{w'} & B' \end{array}$$

w' is a weak equivalence as soon as w is.

Lemma 1.2. *A model category is left proper if and only if each cofibration is an h -cofibration.*

Proof. If in diagram (2) above \mathcal{E} is left proper, f is a cofibration and w a weak equivalence, then w' is a weak equivalence as well, thus f is an h -cofibration. Conversely, if every cofibration is an h -cofibration, then (2) shows that weak equivalences are stable under pushout along cofibrations, so that \mathcal{E} is left proper. \square

Lemma 1.3. *The class of h -cofibrations is closed under composition, cobase change, retract, and under formation of finite coproducts.*

Proof. Closedness under composition, cobase change and retract follows immediately from the definition. For closedness under finite coproducts, it is enough to show that for each h -cofibration $f : X \rightarrow Y$ and each object Z , the coproduct $f \sqcup 1_Z$ is an h -cofibration. This follows from the fact that the commutative square

$$\begin{array}{ccc} X & \longrightarrow & X \sqcup Z \\ f \downarrow & & \downarrow f \sqcup 1_Z \\ Y & \longrightarrow & Y \sqcup Z \end{array}$$

is a pushout. \square

Lemma 1.4. –

- (a) *An object Z is h -cofibrant if and only if $-\sqcup Z$ preserves weak equivalences.*
- (b) *The class of weak equivalences is closed under finite coproducts if and only if all objects of the model category are h -cofibrant.*
- (c) *If all objects of a left proper model category are h -cofibrant then the class of weak equivalences is closed under arbitrary coproducts.*

Proof. The first statement expresses the fact that pushout along the map from an initial object to Z is the same as taking the coproduct with Z . The second statement follows from the first and from the identity $f \sqcup g = (1_{\text{codomain}(f)} \sqcup g) \circ (f \sqcup 1_{\text{domain}(g)})$. The last statement follows from the second and the fact that any coproduct can be calculated as a filtered colimit of finite coproducts with structure maps being coproduct injections. In a left proper category such a filtered colimit is a homotopy colimit (see Proposition 17.9.3 in [29]), hence preserves weak equivalences. \square

The following proposition gives several useful characterisations of h -cofibrations in left proper model categories. Left properness is essential here because homotopy pushouts are easier to recognise in left proper model categories than in general model categories. For instance, in a left proper model category any pushout along a cofibration is a homotopy pushout, which is not the case in general model categories.

A weak equivalence w in X/\mathcal{E} is called a *cofiber equivalence* if for each morphism $g : X \rightarrow B$ the functor $g_! : X/\mathcal{E} \rightarrow B/\mathcal{E}$ takes w to a weak equivalence in B/\mathcal{E} .

Proposition 1.5. *In a left proper model category \mathcal{E} , the following four properties of a morphism $f : X \rightarrow Y$ are equivalent:*

- (i) *f is an h -cofibration;*
- (ii) *every pushout along f is a homotopy pushout;*
- (iii) *for every factorisation of f into a cofibration followed by a weak equivalence, the weak equivalence is a cofiber equivalence;*
- (iv) *there exists a factorisation of f into a cofibration followed by a cofiber equivalence.*

Proof. (i) \implies (ii) For a given outer pushout rectangle like in diagram (2) above with an h -cofibration f , factor the given map $X \rightarrow B$ as a cofibration $X \rightarrow A$ followed by a weak equivalence $w : A \rightarrow B$, and define $Y \rightarrow A'$ as the pushout of $X \rightarrow A$ along f . Since the right square is then a pushout, $w' : A' \rightarrow B'$ is a weak equivalence, whence the outer rectangle is a homotopy pushout.

(ii) \implies (iii) The pushout

$$\begin{array}{ccc} X & \xrightarrow{g} & B \\ f \downarrow & & \downarrow \\ Y & \longrightarrow & B' \end{array}$$

is factored vertically according to a factorisation of f into a cofibration $X \rightarrow Z$ followed by a weak equivalence $v : Z \rightarrow Y$:

$$\begin{array}{ccc} X & \xrightarrow{g} & B \\ \downarrow & & \downarrow \\ Z & \longrightarrow & Z' \\ \downarrow v & & \downarrow v' \\ Y & \longrightarrow & B. \end{array}$$

Since the outer rectangle is a homotopy pushout by assumption, v' is a weak equivalence, whence v is a cofiber equivalence.

(iii) \implies (iv) This is obvious.

(iv) \implies (i) Consider a commutative diagram like in (2) above, and factor f into a cofibration $X \rightarrow Z$ followed by a cofiber equivalence. This induces the following commuting diagram of pushout squares

$$\begin{array}{ccccc} X & \longrightarrow & A & \xrightarrow{w} & B \\ \downarrow & & \downarrow & & \downarrow \\ Z & \longrightarrow & W & \xrightarrow{w''} & Z' \\ \downarrow v & & \downarrow v'' & & \downarrow v' \\ Y & \longrightarrow & A' & \xrightarrow{w'} & B' \end{array}$$

in which v'' and v' are weak equivalences, and w'' is a weak equivalence by left properness of \mathcal{E} . Therefore, the 2-out-of-3 property of the class of weak equivalences implies that w' is a weak equivalence as well, and hence f is an h -cofibration as required. \square

A weak equivalence which is an h -cofibration will be called a *trivial h -cofibration*. A weak equivalence which remains a weak equivalence under any cobase change will be called *couniversal*. For instance, trivial cofibrations are couniversal weak equivalences.

Lemma 1.6. *In general, couniversal weak equivalences are trivial h -cofibrations. In a left proper model category, trivial h -cofibrations are couniversal weak equivalences.*

Proof. The 2-out-of-3 property of the class of weak equivalences implies that couniversal weak equivalences are trivial h -cofibrations. In a left proper model category, the pushout of a trivial h -cofibration is again a trivial h -cofibration by 1.5(ii) and the general fact that weak equivalences are preserved under homotopy pushout. \square

Definition 1.7. *A model category is called h -monoidal if it is a monoidal model category [30] such that for each (trivial) cofibration $f : X \rightarrow Y$ and each object Z , the tensor product $f \otimes 1_Z : X \otimes Z \rightarrow Y \otimes Z$ is a (trivial) h -cofibration.*

It is called strongly h -monoidal if moreover the class of weak equivalences is closed under tensor product.

In particular, each cofibration is an h -cofibration so that, by Lemma 1.2, h -monoidal model categories are *left proper*. Moreover, in virtue of Lemma 1.6, the

condition on trivial cofibrations can be considered as a weak form of the *monoid axiom* of Schwede-Shiely [46], cf. Proposition 2.5 and Corollary 2.6 below.

Lemma 1.8. *For monoidal model categories the following implications hold:*

all objects cofibrant \implies strongly h -monoidal \implies h -monoidal \implies left proper.

Proof. For the first implication, it suffices to observe that, by a well-known argument of Rezk, if all objects are cofibrant then the model structure is left proper, i.e. (by 1.2) cofibrations are h -cofibrations. Moreover, the pushout-product axiom implies that tensoring a (trivial) cofibration with an arbitrary object yields again a (trivial) cofibration. Therefore, the model structure is h -monoidal. The class of weak equivalences is closed under tensor product, since by Brown's Lemma (if all objects are cofibrant) each weak equivalence factors as a trivial cofibration followed by a retraction of a trivial cofibration. The other two implications are obvious. \square

It is in general difficult to describe explicitly the class of h -cofibrations of a model category. The following three propositions are useful since they are applicable even if such an explicit description is unavailable.

Proposition 1.9. *Let \mathcal{E} be a closed symmetric monoidal category with two model structures, called resp. injective and projective, and sharing the same class of weak equivalences. We assume that the following three properties hold:*

- *the projective model structure is a monoidal model structure;*
- *the injective model structure is left proper;*
- *tensoring a (trivial) cofibration of the projective model structure with an arbitrary object yields a (trivial) cofibration of the injective model structure.*

Then the projective model structure is h -monoidal.

Proof. Observe that the notion of h -cofibration only depends on the class of weak equivalences, hence both model structures have the same class of h -cofibrations. The statement then follows directly from Lemma 1.2. \square

Proposition 1.10. *Let \mathcal{E} be a symmetric monoidal category with two monoidal model structures such that each cofibration (resp. weak equivalence, resp. fibration) of the first model structure is an h -cofibration (resp. weak equivalence, resp. fibration) of the second. If all object of the first model structure are cofibrant then both model structures are h -monoidal.*

Proof. Since all objects of the first model structure are cofibrant, the first model structure is (strongly) h -monoidal by Lemma 1.8. Since the trivial fibrations of the first structure are among the trivial fibrations of the second, the cofibrations of the second are among the cofibrations of the first. The latter class is closed under tensor product and contained in the class of h -cofibrations of the second structure. This yields the first half of h -monoidality for the second model structure. Similarly, since the fibrations of the first model structure are among the fibrations of the second, the trivial cofibrations of the second are among the trivial cofibrations of the first. The latter class is closed under tensor product and contained in the class of trivial h -cofibrations of the second model structure. This shows that the second half of h -monoidality holds for the second model structure as well. \square

Proposition 1.11. *Let \mathcal{E} be a monoidal model category in which all objects are fibrant. Then \mathcal{E} is h -monoidal provided the internal hom of \mathcal{E} detects weak equivalences in the following sense: a map $f : X \rightarrow Y$ is a weak equivalence whenever $\underline{\mathcal{E}}(f, W)$ is a weak equivalence for all objects W .*

Proof. Let $f : X \rightarrow Y$ be a cofibration. We have to show that in

$$\begin{array}{ccccc} X \otimes Z & \longrightarrow & A & \xrightarrow{w} & B \\ f \otimes Z \downarrow & & \downarrow & & \downarrow \\ Y \otimes Z & \longrightarrow & A' & \xrightarrow{w'} & B' \end{array}$$

w' is a weak equivalence if w is. For this it suffices to show that $\underline{\mathcal{E}}(w', W)$ is a weak equivalence for all W , which follows from the pushout-product axiom and the hom-tensor adjunction. If f is a trivial cofibration then $\underline{\mathcal{E}}(f \otimes Z, W) \cong \underline{\mathcal{E}}(f, \underline{\mathcal{E}}(Z, W))$ is a trivial fibration for each object W . Hence $f \otimes Z$ is a weak equivalence. \square

Examples 1.12. Below a list of frequently used monoidal model categories in which all objects are cofibrant. By Lemma 1.8 they are thus strongly h -monoidal.

- simplicial sets;
- small categories with the folklore model structure, cf. [34, 12];
- Rezk's model for (∞, n) -categories [43];
- compactly generated spaces with Strøm's model structure;
- chain complexes over a field with the projective model structure.

There are strongly h -monoidal model categories in which not all objects are cofibrant, e.g.

- compactly generated spaces with Quillen's model structure (use 1.11 or Corollary 1.13 below);
- small 2-categories (or 2-groupoids) with the Gray tensor product (use 1.11, cf. [34, 12]).

Corollary 1.14 treats two examples of h -monoidal model categories which are not strongly h -monoidal.

Corollary 1.13. *The category of compactly generated topological spaces is strongly h -monoidal with respect to Strøm's and Quillen's model structures.*

Proof. Recall that in Strøm's model structure the weak equivalences and fibrations are homotopy equivalences and Hurewicz fibrations respectively; the corresponding classes in Quillen's model structure are weak homotopy equivalences and Serre fibrations. These classes verify the inclusion relations required by Proposition 1.10. The cofibrations of Strøm's model structure are the closed cofibrations in the topologist's classical sense. It is known (though not well-known) that closed cofibrations are h -cofibrations for Quillen's model structure. In Strøm's model structure all objects are cofibrant so that it is strongly h -monoidal by Lemma 1.8. Proposition 1.10 implies that Quillen's model structure is h -monoidal. It is strongly h -monoidal since the product of two weak homotopy equivalences is again a weak homotopy equivalence. \square

Corollary 1.14. *The following two examples are h -monoidal model categories:*

- the category of chain complexes over a commutative ring with the projective model structure;
- the category of symmetric spectra (in simplicial sets) with the stable projective model structure.

Proof. We use in both cases Proposition 1.9. Recall that the cofibrations of the injective (resp. projective) model structure on chain complexes are the monomorphisms (resp. monomorphisms with degreewise projective quotient). In particular, a projective cofibration $f_\bullet : X_\bullet \rightarrow Y_\bullet$ is degreewise split so that $f_\bullet \otimes Z_\bullet$ is degreewise a monomorphism, and hence a cofibration in the injective model structure. If f_\bullet is trivial (i.e. a quasi-isomorphism), its degreewise projective quotient Y_\bullet/X_\bullet is acyclic, and hence contractible. Therefore $(Y_\bullet/X_\bullet) \otimes Z_\bullet$ is contractible as well, and hence $f_\bullet \otimes Z_\bullet$ is trivial as required. The statement about symmetric spectra follows by an analogous argument from Proposition III.1.11i and Lemma III.1.4 of Schwede's book project [47]. \square

In the examples we have treated so far, all objects were actually *h-cofibrant*. The following proposition shows why this must be the case. This is actually the only place in this article where we explicitly use the *unit axiom* of Hovey [30].

Proposition 1.15. *In a left proper monoidal model category, the unit is necessarily h-cofibrant. In an h-monoidal model category, all objects are h-cofibrant, and hence the class of weak equivalences is closed under arbitrary coproducts.*

Proof. We shall use Lemma 1.4i for recognising *h-cofibrant* objects. The unit axiom requires the existence of a *cofibrant replacement* $Q(e) \rightarrow e$ for the *unit* e , which remains a weak equivalence after tensoring with arbitrary objects. For each object X of \mathcal{E} , we thus have a weak equivalence $Q(e) \otimes (e \sqcup X) \rightarrow e \sqcup X$. Since the tensor commutes with coproducts, this weak equivalence can be rewritten as

$$Q(e) \sqcup (Q(e) \otimes X) \rightarrow Q(e) \sqcup X \rightarrow e \sqcup X$$

where the first map is the coproduct of $Q(e)$ with $Q(e) \otimes X \rightarrow X$. Therefore, if the monoidal model category is left proper, and hence $Q(e)$ is *h-cofibrant* by Lemma 1.2, then the first map above is a weak equivalence. By the 2-of-3 property of weak equivalences, the second map $Q(e) \sqcup X \rightarrow e \sqcup X$ is a weak equivalence as well. But then, for each weak equivalence $X \rightarrow Y$, the commutative diagram

$$\begin{array}{ccc} Q(e) \sqcup X & \longrightarrow & Q(e) \sqcup Y \\ \downarrow & & \downarrow \\ e \sqcup X & \longrightarrow & e \sqcup Y \end{array}$$

implies that $e \sqcup X \rightarrow e \sqcup Y$ is a weak equivalence, which shows that e is *h-cofibrant*.

In an *h-monoidal* model category we have the stronger property that $Q(e) \otimes Z$ is *h-cofibrant* for each object Z . Therefore, factoring the weak equivalence

$$Q(e) \otimes (Z \sqcup X) = (Q(e) \otimes Z) \sqcup (Q(e) \otimes X) \rightarrow Z \sqcup X$$

through $(Q(e) \otimes Z) \sqcup X$ yields a weak equivalence $(Q(e) \otimes Z) \sqcup X \rightarrow Z \sqcup X$ for all objects Z and X . This implies as above that all objects Z are *h-cofibrant*. \square

The following lemma is also useful to retain:

Lemma 1.16. *In an h-monoidal model category, tensoring a weak equivalence between cofibrant objects with an arbitrary object yields again a weak equivalence.*

Proof. By Brown's Lemma, a weak equivalence between cofibrant objects factors as a trivial cofibration followed by a retraction of a trivial cofibration. Both factors yield a weak equivalence when tensored with an arbitrary object. \square

2. ADMISSIBLE MONADS ON COMPACTLY GENERATED MODEL CATEGORIES

It is well-known that the class of (trivial) cofibrations in an arbitrary model category is closed under cobase change, transfinite composition and retract. Classes of morphisms with these three closure properties will be called *saturated*.

Definition 2.1. *With respect to saturated class of morphisms K in a model category \mathcal{E} , the class W of weak equivalences of \mathcal{E} is called K -perfect if W is closed under filtered colimits along morphisms in K .*

Remark 2.2. By Hovey's argument [30, 7.4.2] a sufficient condition for the K -perfectness of the class of weak equivalences is the existence of a *generating set of cofibrations* whose domain and codomain are *finite* with respect to K .

Lemma 2.3. *If the class W of weak equivalences is K -perfect then the intersection $W \cap K$ is closed under transfinite composition.*

Proof. Any transfinite composition of maps can be identified with the colimit of a natural transformation from a constant diagram to the given sequence of maps. If the given maps belong to $W \cap K$ this colimit is a filtered colimit of weak equivalences along morphisms in K . By assumption such a colimit is a weak equivalence. \square

We shall say that a class of morphisms is *monoidally saturated* if it is saturated and moreover closed under tensoring with *arbitrary objects* of the monoidal model category. Accordingly, the *monoidal saturation* of a class K is the least monoidally saturated class containing K . For instance, in virtue of the pushout-product axiom, the class of (trivial) cofibrations of a monoidal model category is monoidally saturated whenever all objects of the model category are cofibrant.

We are mainly interested in the monoidal saturation of the class of cofibrations. This monoidal saturation will be denoted I^\otimes since it suffices to monoidally saturate a generating set of cofibrations which traditionally is denoted I . For brevity we shall call \otimes -*cofibration* any morphism in I^\otimes . An object will be called \otimes -*small* (resp. \otimes -*finite*) if it is small (resp. finite) with respect to I^\otimes . The class of weak equivalences will be called \otimes -*perfect* if it is I^\otimes -perfect.

Definition 2.4 (cf. [12]). *A model category is called K -compactly generated if it is cofibrantly generated, its class of weak equivalences is K -perfect, and each object is small with respect to K .*

A monoidal model category is called compactly generated, if the underlying model category is I^\otimes -compactly generated.

For instance, any monoidal model category whose underlying model category is *combinatorial*, and whose class of weak equivalences is *closed under filtered colimits*, is an example of a compactly generated monoidal model category. The majority of our examples are of this kind. However, compactly generated topological spaces form a monoidal model category which is neither combinatorial nor does it have a class of weak equivalences which is closed under filtered colimits. Yet, every compactly generated space is \otimes -small, and the class of weak equivalences is \otimes -perfect,

hence the monoidal model category of compactly generated spaces is compactly generated in the aforementioned model-theoretical sense, cf. [30, 12].

Proposition 2.5. *In any compactly generated h -monoidal model category, the monoid axiom of Schwede-Shipley holds and each \otimes -cofibration is an h -cofibration.*

Proof. The monoid axiom of Schwede-Shipley [46] requires the monoidal saturation of the class of trivial cofibrations to stay with the class of weak equivalences. In a cofibrantly generated monoidal model category this monoidal saturation can be constructed by choosing a generating set J for the trivial cofibrations, and saturating the class $\{f \otimes 1_Z \mid f \in J, Z \in \text{Ob}\mathcal{E}\}$ under cobase change, transfinite composition and retract. Since, by Lemma 1.6, each $f \otimes 1_Z$ is a couniversal weak equivalence and a \otimes -cofibration, and both classes are closed under cobase change and retract, it remains to be shown that the class of maps, which are simultaneously weak equivalences and \otimes -cofibrations, is closed under transfinite composition. This is precisely Lemma 2.3 for $K = I^\otimes$.

For the second statement, we have to show that the monoidal saturation of the class of cofibrations stays within the class of h -cofibrations. As before, this monoidal saturation can be constructed by choosing a generating set I for the cofibrations, and saturating the class $\{f \otimes 1_Z \mid f \in I, Z \in \text{Ob}\mathcal{E}\}$ under cobase change, transfinite composition and retract. Since each $f \otimes 1_Z$ is an h -cofibration and a \otimes -cofibration, and both classes are closed under cobase change and retract, it remains to be shown that the class of maps, which are simultaneously h -cofibrations and \otimes -cofibrations, is closed under transfinite composition. This follows from the definition of an h -cofibration, since Lemma 2.3 (for $K = I^\otimes$) shows that a vertical transfinite composition of diagrams of the form (2) (all vertical maps being h -cofibrations and \otimes -cofibrations) yields a diagram of the same form (2). \square

Corollary 2.6. *In a monoidal model category with \otimes -perfect class of weak equivalences, the monoid axiom of Schwede-Shipley holds if and only if the tensor product of a trivial cofibration with an arbitrary object is a couniversal weak equivalence.*

Proof. This follows from the argument of first paragraph of the preceding proof. \square

Remark 2.7. The preceding proposition and corollary (together with 1.9, 1.10 or 1.11) may be an efficient tool to establish the monoid axiom and left properness. For instance, Lack's original proofs [34, Theorems 6.3 and 7.7] of these properties for the category of small 2-categories are quite a bit more involved.

2.8. Admissible monads. Recall that a monad T on \mathcal{E} is called *finitary* if T preserves filtered colimits, or what amounts to the same, if the forgetful functor $U_T : \text{Alg}_T \rightarrow \mathcal{E}$ preserves filtered colimits. Here, Alg_T denotes the category of T -algebras and

$$F_T : \mathcal{E} \rightleftarrows \text{Alg}_T : U_T$$

the free-forgetful adjunction. Thus $T = U_T F_T$ and $F_T(X) = (TX, \mu_X)$ where $\mu : T^2 \rightarrow T$ is the multiplication of the monad T .

Definition 2.9. *Let \mathcal{E} be a model category, W its class of weak equivalences, and K be an arbitrary saturated class in \mathcal{E} . A monad T on \mathcal{E} is said to be K -admissible if for each cofibration (resp. trivial cofibration) $u : X \rightarrow Y$ and each map of*

T -algebras $\alpha : F_T(X) \rightarrow R$, the pushout in Alg_T

$$(3) \quad \begin{array}{ccc} F_T(X) & \xrightarrow{\alpha} & R \\ F_T(u) \downarrow & \lrcorner & \downarrow u_\alpha \\ F_T(Y) & \longrightarrow & R[u, \alpha] \end{array}$$

yields a T -algebra map $u_\alpha : R \rightarrow R[u, \alpha]$ whose underlying map $U_T(u_\alpha)$ belongs to K (resp. to $W \cap K$).

Recall that a map of free T -algebras $F_T(u) : F_T(X) \rightarrow F_T(Y)$ is an h -cofibration if for any diagram of pushouts in Alg_T

$$(4) \quad \begin{array}{ccccccc} F_T(X) & \xrightarrow{\alpha} & R & \xrightarrow{f} & S \\ F_T(u) \downarrow & \lrcorner & \downarrow & \lrcorner & \downarrow \\ F_T(Y) & \longrightarrow & R[u, \alpha] & \longrightarrow & S[u, f\alpha] \end{array}$$

in which $f : R \rightarrow S$ is a weak equivalence, the induced map $R[u, \alpha] \rightarrow S[u, f\alpha]$ is again a weak equivalence. We shall say that $F_T(u)$ is a *relative h -cofibration* if the latter preservation property only holds for those $f : R \rightarrow S$ for which $U_T(R)$ and $U_T(S)$ are cofibrant in \mathcal{E} .

Definition 2.10. A model structure on T -algebras will be called *relatively left proper* if weak equivalences $f : R \rightarrow S$, for which $U_T(R)$ and $U_T(S)$ are cofibrant in \mathcal{E} , are closed under cobase change along cofibrations of T -algebras.

Theorem 2.11. For any finitary K -admissible monad T on a K -compactly generated model category \mathcal{E} , the category of T -algebras admits a transferred model structure. This model structure is (relatively) left proper if and only if the free T -algebra functor takes cofibrations in \mathcal{E} to (relative) h -cofibrations in Alg_T .

Proof. By definition of a transfer, a map of T -algebras f is defined to be a weak equivalence (resp. fibration) precisely when $U_T(f)$ is a weak equivalence (resp. fibration) in \mathcal{E} . Cofibrations of T -algebras are defined by the left lifting property with respect to trivial fibrations. In order to show that these three classes define a model structure on Alg_T , the main difficulty consists in proving the existence of cofibration/trivial fibration (resp. trivial cofibration/fibration) factorisations. For this we apply Quillen's small object argument to the image $F_T(I)$ (resp. $F_T(J)$) of a generating set I (resp. J) for the cofibrations (resp. trivial cofibrations) of \mathcal{E} . The following two points have to be shown:

- (i) The domains of the maps in $F_T(I)$ (resp. $F_T(J)$) are small with respect to the saturation of $F_T(I)$ (resp. $F_T(J)$) under cobase change and transfinite composition in Alg_T ;
- (ii) The saturation of $F_T(J)$ under cobase change and transfinite composition in Alg_T stays within the class of weak equivalences.

Since the forgetful functor U_T preserves filtered colimits, an adjunction argument and the K -smallness of the objects of \mathcal{E} yield (i). Moreover, Lemma 2.3 and the K -perfectness of the weak equivalences in \mathcal{E} yield (ii).

If the transferred model structure on Alg_T is (relatively) left proper then the left Quillen functor F_T takes cofibrations in \mathcal{E} to (relative) h -cofibrations in Alg_T

by Lemma 1.2. Conversely, assume that $F_T(u)$ is a (relative) h -cofibration for each generating cofibration u . Note first that the forgetful functor U_T preserves transfinite compositions since it preserves filtered colimits. It follows then from the K -perfectness of the class of weak equivalences and the K -admissibility of T that cobase change along a transfinite composition of free T -algebra extensions of the form $R \rightarrow R[u, \alpha]$ preserves weak equivalences (between T -algebras with underlying cofibrant domain and codomain). But any cofibration in Alg_T is retract of such a transfinite composition. Thus, Alg_T is (relatively) left proper. \square

Proposition 2.12. *The free T -algebra functor takes cofibrations to relative h -cofibrations if it takes cofibrations with cofibrant domain to relative h -cofibrations.*

Proof. Suppose that $u : X \rightarrow Y$ is a cofibration. We have to show that for a weak equivalence $f : R \rightarrow S$ with cofibrant underlying objects $U_T(R), U_T(S)$, the morphism $R[u, \alpha] \rightarrow S[u, f\alpha]$ in the diagram (4) is a weak equivalence. Let $\alpha' : X \rightarrow U_T(R)$ be the composite

$$X \xrightarrow{\epsilon} U_T F_T(X) \xrightarrow{U_T(\alpha)} U_T(R)$$

and consider the following pushout in \mathcal{E} :

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & U_T(R) \\ u \downarrow & & \downarrow v \\ Y & \longrightarrow & P \end{array}$$

The given map α factors as

$$F_T(X) \xrightarrow{F_T(\alpha')} F_T U_T(R) \xrightarrow{k} R$$

where k is the structure map of the T -algebra R . Therefore, by the universal property of pushouts, the right-hand square of the following commutative diagram

$$\begin{array}{ccccc} F_T(X) & \xrightarrow{F_T(\alpha')} & F_T U_T(R) & \xrightarrow{k} & R \\ F_T(u) \downarrow & & \downarrow & & \downarrow \\ F_T(Y) & \longrightarrow & F_T(P) & \longrightarrow & R[u, \alpha] \end{array}$$

is a pushout. Hence, we get the following pushout diagram in Alg_T :

$$\begin{array}{ccccc} F_T U_T(R) & \xrightarrow{k} & R & \xrightarrow{f} & S \\ F_T(v) \downarrow & & \downarrow & & \downarrow \\ F_T(P) & \longrightarrow & R[u, \alpha] & \longrightarrow & S[u, f\alpha] \end{array}$$

Since v is a cofibration with cofibrant domain, $F_T(v)$ is a relative h -cofibration by assumption, so that $R[u, \alpha] \rightarrow S[u, f\alpha]$ is a weak equivalence as required. \square

Definition 2.13. *A monad T is K -adequate if the underlying map of any free T -algebra extension $u_\alpha : R \rightarrow R[u, \alpha]$ admits a functorial factorisation*

$$U_T(R) = R[u]^{(0)} \rightarrow R[u]^{(1)} \rightarrow \dots \rightarrow R[u]^{(n)} \rightarrow \dots \rightarrow \text{colim}_n R[u]^{(n)} = U_T(R[u, \alpha]);$$

such that for a cofibration (resp. trivial cofibration) u , each map of the sequence belongs to K (resp. $W \cap K$), and moreover for a weak equivalence $f : R \rightarrow S$, the induced morphisms $R[u]^{(n)} \rightarrow S[u]^{(n)}$ are weak equivalences for all $n \geq 0$.

The monad T is relatively K -adequate if the last property only holds if u is a cofibration with cofibrant domain and $f : R \rightarrow S$ is a weak equivalence with cofibrant underlying objects $U_T(R)$ and $U_T(S)$.

Theorem 2.14. *Any finitary (relatively) K -adequate monad T on a K -compactly generated model category \mathcal{E} is K -admissible, and the associated free T -algebra functor takes cofibrations to (relative) h -cofibrations. Hence, the category of T -algebras has a transferred model structure which is (relatively) left proper.*

Proof. The second statement follows from the first and from Theorem 2.11. K -admissibility, cf. (3), follows from Lemma 2.3. It remains to be shown that, given a cofibration $u : X \rightarrow Y$ and a weak equivalence $f : R \rightarrow S$ of T -algebras, the induced morphism $R[u, \alpha] \rightarrow S[u, f\alpha]$ in diagram (4) is a weak equivalence.

For the relative version we assume that $U_T(R)$ and $U_T(S)$ are cofibrant and that u has a cofibrant domain (see Proposition 2.12). By functoriality of factorisations the underlying map of this morphism is a sequential colimit of a ladder in \mathcal{E}

$$\begin{array}{ccccccc} R[u]^{(0)} & \rightarrow & R[u]^{(1)} & \rightarrow & \cdots & \rightarrow & R[u]^{(n)} & \rightarrow & \cdots & \rightarrow & \operatorname{colim}_n R[u]^{(n)} = U_T(R[u, \alpha]) \\ \downarrow & & \downarrow & & & & \downarrow & & & & \downarrow \\ S[u]^{(0)} & \rightarrow & S[u]^{(1)} & \rightarrow & \cdots & \rightarrow & S[u]^{(n)} & \rightarrow & \cdots & \rightarrow & \operatorname{colim}_n S[u]^{(n)} = U_T(S[u, f\alpha]) \end{array}$$

in which the vertical maps are weak equivalences and the horizontal maps belong to K . Since \mathcal{E} is K -compactly generated this colimit is a weak equivalence. \square

3. MONOIDS IN h -MONOIDAL MODEL CATEGORIES

This section presents the main result of Schwede-Shipley [46] concerning the existence of a model structure on monoids if the monoid axiom holds. We add a discussion of left properness of the transferred model structure.

Recall that I^\otimes denotes the monoidal saturation of the class of cofibrations, and that any morphism in I^\otimes is called a \otimes -cofibration. Accordingly, we say \otimes -admissible (resp. \otimes -adequate) instead of I^\otimes -admissible (resp. I^\otimes -adequate).

Theorem 3.1 (cf. Schwede-Shipley [46]). *For any compactly generated monoidal model category \mathcal{E} the free monoid monad T on \mathcal{E} is :*

- (a) *relatively \otimes -adequate if the monoid axiom holds;*
- (b) *\otimes -adequate if \mathcal{E} is strongly h -monoidal.*

And hence

- (a') *there is a relatively left proper transferred model structure on monoids if the monoid axiom holds;*
- (b') *the model structure on monoids is left proper if \mathcal{E} is strongly h -monoidal.*

Proof. (a'), (b') follow from (a), (b) and Theorem 2.14.

Let R be a monoid in \mathcal{E} , and let $u : Y_0 \rightarrow Y_1$ be a map in \mathcal{E} equipped with a map of monoids $F_T(Y_0) \rightarrow R$. We shall exhibit the pushout in the category of monoids as a sequential colimit in \mathcal{E} .

Let $R[u]^{(0)} = R$ and define inductively $R[u]^{(n)}$ by the following pushout

$$\begin{array}{ccc} Y_-^{(n)} & \longrightarrow & R[u]^{(n-1)} \\ \downarrow & \lrcorner & \downarrow \\ Y^{(n)} & \longrightarrow & R[u]^{(n)} \end{array}$$

where

$$Y^{(n)} = R \otimes \overbrace{Y_1 \otimes R \otimes \cdots \otimes Y_1 \otimes R}^n$$

and $Y_-^{(n)}$ is the colimit of a diagram over a punctured n -cube $\{0, 1\}^n - \{(1, \dots, 1)\}$ in which the vertex (i_1, \dots, i_n) takes the value

$$R \otimes Y_{i_1} \otimes \cdots \otimes R \otimes Y_{i_n} \otimes R$$

and the edge-maps are induced by u . The map $Y_-^{(n)} \rightarrow Y^{(n)}$ is the comparison map from the colimit of this diagram to the value at $(1, \dots, 1)$ of the extended diagram on the whole n -cube. The map $Y_-^{(n)} \rightarrow R[u]^{(n-1)}$ is defined inductively, using the fact that the construction of $R[u]^{(n-1)}$ involves $n - 1$ tensor factors only.

Since the tensor $-\otimes-$ commutes with pushouts in both variables, there are canonical maps of $R[u]^{(p)} \otimes R[u]^{(q)} \rightarrow R[u]^{(p+q)}$. Since the tensor $-\otimes-$ commutes with sequential colimits in both variables, these maps induce the structure of a monoid on the colimit $\text{colim}_n R[u]^{(n)}$. It has been checked in [46] that this monoid has indeed the universal property of $R[u]$.

We shall now prove that, for each $n > 0$, the map $R[u]^{(n-1)} \rightarrow R[u]^{(n)}$ is a \otimes -cofibration (resp. weak equivalence) whenever u is a cofibration (resp. trivial cofibration). The considered map derives from $Y_-^{(n)} \rightarrow Y^{(n)}$ through a cobase change. Collecting all tensor factors R , the map $Y_-^{(n)} \rightarrow Y^{(n)}$ may be identified with an iterated pushout-product map along u , tensored with $R^{\otimes n+1}$. Therefore, $Y_-^{(n)} \rightarrow Y^{(n)}$ as well as $R[u]^{(n-1)} \rightarrow R[u]^{(n)}$ are \otimes -cofibrations. If u is a trivial cofibration, the iterated product-map is a trivial cofibration, and the monoid axiom implies that its tensor product with $R^{\otimes n+1}$ is a couniversal weak equivalence; thus $R[u]^{(n-1)} \rightarrow R[u]^{(n)}$ is a weak equivalence. Since \mathcal{E} is compactly generated, this yields \otimes -admissibility of the monad by Lemma 2.3. We will address relative \otimes -adequateness after the proof of (b).

For the proof of (b), let us assume that we are given a weak equivalence $f : R \rightarrow S$ between T -algebras, and that \mathcal{E} is strongly h -monoidal. The functoriality of the construction gives rise, for each $n > 0$, to a commutative cube in \mathcal{E}

$$(5) \quad \begin{array}{ccccc} & & Z_-^{(n)} & \xrightarrow{\quad} & S[u]^{(n-1)} \\ & \nearrow & \downarrow & \nearrow & \downarrow \\ Y_-^{(n)} & \xrightarrow{\quad} & R[u]^{(n-1)} & & \\ \downarrow & & \downarrow & & \downarrow \\ & \nearrow & Z^{(n)} & \xrightarrow{\quad} & S[u]^{(n)} \\ Y^{(n)} & \xrightarrow{\quad} & R[u]^{(n)} & & \end{array}$$

in which $Z_-^{(n)} \rightarrow Z^{(n)}$ is defined like $Y_-^{(n)} \rightarrow Y^{(n)}$ just replacing R with S .

Front and back square of the cube are pushouts, actually homotopy pushouts, since $Y_-^{(n)} \rightarrow Y^{(n)}$ and $Z_-^{(n)} \rightarrow Z^{(n)}$ are h -cofibrations by h -monoidality of \mathcal{E} . The natural transformation from front to back square is induced by tensor powers of $f : R \rightarrow S$. By induction, it suffices now to show that $R[u]^{(n)} \rightarrow S[u]^{(n)}$ is a weak equivalence whenever $R[u]^{(n-1)} \rightarrow S[u]^{(n-1)}$ is.

Since \mathcal{E} is strongly h -monoidal, the tensor power $f^{\otimes n+1} : R^{\otimes n+1} \rightarrow S^{\otimes n+1}$ is again a weak equivalence. Hence, for any vertex (i_1, \dots, i_n) of the n -cube, the map

$$(6) \quad R \otimes Y_{i_1} \otimes \dots \otimes R \otimes Y_{i_n} \otimes R \rightarrow S \otimes Y_{i_1} \otimes \dots \otimes S \otimes Y_{i_n} \otimes S$$

is a weak equivalence. In particular, the map $Y^{(n)} \rightarrow Z^{(n)}$ is a weak equivalence.

Moreover, since u is a cofibration and \mathcal{E} is h -monoidal, each morphism inside the defining punctured n -cubes is an h -cofibration. Therefore, $Y_-^{(n)}$ and $Z_-^{(n)}$ are homotopy pushouts and the induced map $Y_-^{(n)} \rightarrow Z_-^{(n)}$ is a weak equivalence as well. Hence, the left hand square of (5) is also a homotopy pushout. It follows then from known properties of homotopy pushouts in left proper model categories (cf. [26]) that the right hand square is a homotopy pushout as well, so that $R[u]^{(n)} \rightarrow S[u]^{(n)}$ is a weak equivalence as required.

Let us come back to the relative \otimes -adequateness of free monoid monad and assume that $U_T(R)$ and $U_T(S)$ are cofibrant in \mathcal{E} , and that u has a cofibrant domain. It follows then from the pushout-product axiom that all objects of the cube (5) are cofibrant, that the two left vertical maps are cofibrations, and that the two left horizontal maps are weak equivalences. This implies (by a well-known gluing lemma) that $R[u]^{(n)} \rightarrow S[u]^{(n)}$ is a weak equivalence too, as required. \square

4. DIAGRAM CATEGORIES AND DAY CONVOLUTION

As a first application of our methods we observe that the class of compactly generated (strongly) h -monoidal categories \mathcal{E} is closed under taking diagram categories over a small \mathcal{E} -enriched category \mathbb{C} . More precisely, let \mathcal{E} be a monoidal model category and \mathbb{C} be a small \mathcal{E} -enriched category. Let $[\mathbb{C}, \mathcal{E}]$ be the category of \mathcal{E} -enriched functors and \mathcal{E} -natural transformations. Let \mathbb{C}_0 be the set of objects of \mathbb{C} , considered as a discrete \mathcal{E} -category. We have an inclusion \mathcal{E} -functor $i : \mathbb{C}_0 \rightarrow \mathbb{C}$. The category $[\mathbb{C}_0, \mathcal{E}] \cong \mathcal{E}^{\mathbb{C}_0}$ has an obvious product model structure.

There is a monad $i^* i_!$ on $[\mathbb{C}_0, \mathcal{E}]$ where i^* denotes the restriction functor and $i_!$ its left adjoint. The restriction functor i^* is monadic, and the *projective model structure* on $[\mathbb{C}, \mathcal{E}]$ is by definition the model structure which is transferred from $[\mathbb{C}_0, \mathcal{E}]$ along the adjunction $i_! : [\mathbb{C}_0, \mathcal{E}] \rightleftarrows [\mathbb{C}, \mathcal{E}] : i^*$ if such a transfer exists.

We shall call an object of \mathcal{E} *discrete* if it is a coproduct of copies of the unit of \mathcal{E} . Clearly, any tensor product of discrete objects is again discrete.

Theorem 4.1. *Let \mathcal{E} be a compactly generated monoidal model category, and let \mathbb{C} be a small \mathcal{E} -enriched category. Then the projective model structure on $[\mathbb{C}, \mathcal{E}]$ exists in each of the following three cases:*

- (i) *all hom-objects of \mathbb{C} are discrete in \mathcal{E} ;*
- (ii) *all hom-objects of \mathbb{C} are cofibrant in \mathcal{E} ;*
- (iii) *the monoid axiom holds in \mathcal{E} .*

The projective model structure on $[\mathbb{C}, \mathcal{E}]$ is left proper if either \mathcal{E} is h -monoidal (and hence (iii) holds), or if \mathcal{E} is just left proper, but (i) or (ii) holds. If moreover \mathbb{C} is equipped with a symmetric monoidal structure, then $[\mathbb{C}, \mathcal{E}]$ is a compactly generated monoidal model category with respect to Day's convolution product, and

- (a) the monoid axiom holds in $[\mathbb{C}, \mathcal{E}]$ whenever it holds in \mathcal{E} ;
- (b) $[\mathbb{C}, \mathcal{E}]$ is (strongly) h -monoidal whenever \mathcal{E} is (strongly) h -monoidal;
- (c) all objects in $[\mathbb{C}, \mathcal{E}]$ are h -cofibrant whenever all objects in \mathcal{E} are h -cofibrant.

Proof. The existence and left properness of the projective model structure on $[\mathbb{C}, \mathcal{E}]$ will be deduced from Theorem 2.11 if we prove that the monad $i^*i_!$ is K -admissible, where K is the class of pointwise \otimes -cofibrations. Note that class of weak equivalences in $[\mathbb{C}_0, \mathcal{E}]$ is K -perfect. Moreover, all objects of $[\mathbb{C}_0, \mathcal{E}]$ are K -small, so that $[\mathbb{C}_0, \mathcal{E}]$ is a K -compactly generated model category.

Now, for any object X of $[\mathbb{C}_0, \mathcal{E}]$, we have $(i_!X)(a) = \sqcup_{b \in \mathbb{C}_0} \mathbb{C}(b, a) \otimes X(b)$. Let $u : X \rightarrow Y$ be a morphism in $[\mathbb{C}_0, \mathcal{E}]$ and let $\alpha : i_!X \rightarrow R$ be a morphism in $[\mathbb{C}, \mathcal{E}]$. This defines for each $a \in \mathbb{C}_0$ the following pushout

$$\begin{array}{ccc} \prod_{b \in \mathbb{C}} \mathbb{C}(b, a) \otimes X(b) & \xrightarrow{\alpha} & R(a) \\ \downarrow & \lrcorner & \downarrow u \\ \prod_{b \in \mathbb{C}} \mathbb{C}(b, a) \otimes Y(b) & \xrightarrow{\quad} & R[u, \alpha](a) \end{array}$$

in \mathcal{E} . For the K -admissibility of $i^*i_!$ we have to show that the right vertical map is a \otimes -cofibration (resp. weak equivalence) if $u : X \rightarrow Y$ is a cofibration (resp. trivial cofibration). This is obviously the case under assumptions (i) and (ii). For case (iii), note first that a pushout $u : R(a) \rightarrow R[u, \alpha](a)$ like above can be realised as a transfinite composition of pushouts of single maps $\mathbb{C}(b, a) \otimes X(b) \rightarrow \mathbb{C}(b, a) \otimes Y(b)$. For any cofibration u , such a pushout is a \otimes -cofibration, and hence a transfinite composition of them is again a \otimes -cofibration. For a trivial cofibration u , the analogous transfinite composition belongs to the monoidal saturation of the class of trivial cofibrations, and is therefore a weak equivalence under assumption (iii).

If \mathcal{E} is left proper and \mathbb{C} satisfies (i) or (ii) then the left vertical map above is a cofibration, and left properness of \mathcal{E} implies left properness of $[\mathbb{C}, \mathcal{E}]$. Under assumption (iii) and assuming that \mathcal{E} is h -monoidal, Proposition 2.5 shows that the left vertical map above is an h -cofibration which implies left properness of $[\mathbb{C}, \mathcal{E}]$.

From now on we assume that \mathbb{C} is a symmetric monoidal category with tensor

$$\odot : \mathbb{C} \otimes \mathbb{C} \rightarrow \mathbb{C}$$

and we endow $[\mathbb{C}, \mathcal{E}]$ with the Day convolution product. There is an external tensor product $\bar{\otimes} : [\mathbb{C}, \mathcal{E}] \otimes [\mathbb{C}, \mathcal{E}] \rightarrow [\mathbb{C} \otimes \mathbb{C}, \mathcal{E}]$ which is a Quillen functor of two variables with respect to the projective model structures on both sides, cf. Barwick [3]. Left Kan extension along the tensor $\odot : \mathbb{C} \otimes \mathbb{C} \rightarrow \mathbb{C}$ also yields a left Quillen functor $\odot_! : [\mathbb{C} \otimes \mathbb{C}, \mathcal{E}] \rightarrow [\mathbb{C}, \mathcal{E}]$. Therefore, the composite functor

$$-\square- : [\mathbb{C}, \mathcal{E}] \otimes [\mathbb{C}, \mathcal{E}] \xrightarrow{\bar{\otimes}} [\mathbb{C} \otimes \mathbb{C}, \mathcal{E}] \xrightarrow{\odot_!} [\mathbb{C}, \mathcal{E}]$$

which may be identified with the Day convolution product, is a left Quillen functor of two variables, hence $[\mathbb{C}, \mathcal{E}]$ satisfies the pushout-product axiom. The unit axiom for $[\mathbb{C}, \mathcal{E}]$ follows from the unit axiom for \mathcal{E} .

For the compact generation of $[\mathbb{C}, \mathcal{E}]$, note first that the projective model structure on $[\mathbb{C}, \mathcal{E}]$ is K -compactly generated for the saturated class K of pointwise \otimes -cofibrations, since the weak equivalences of $[\mathbb{C}, \mathcal{E}]$ are pointwise weak equivalences, and colimits in $[\mathbb{C}, \mathcal{E}]$ are computed pointwise. Therefore, it suffices to show that each generating cofibration of $[\mathbb{C}, \mathcal{E}]$ belongs to K , and that K is stable under Day convolution $-\square Z$ with an arbitrary object Z . The aforementioned formula for the left adjoint $i_!$ shows that $i_!$ takes cofibrations in $[\mathbb{C}_0, \mathcal{E}]$ to pointwise \otimes -cofibrations in $[\mathbb{C}, \mathcal{E}]$. Observe furthermore that $X\square Z$ is a pointwise retract of $(i_!i^*X)\square(i_!i^*Z)$, and hence $f\square Z$ is a pointwise retract of $(i_!i^*f)\square(i_!i^*Z)$ for any map $f : X \rightarrow Y$ in $[\mathbb{C}, \mathcal{E}]$. The latter morphism evaluated at $c \in \mathbb{C}_0$ is given by

$$(7) \quad \coprod_{a,b} \mathbb{C}(a \odot b, c) \otimes X(a) \otimes Z(b) \rightarrow \coprod_{a,b} \mathbb{C}(a \odot b, c) \otimes Y(a) \otimes Z(b)$$

which is a \otimes -cofibration whenever $f : X \rightarrow Y$ is a pointwise \otimes -cofibration. Hence, the pointwise retract $f\square Z$ is also a pointwise \otimes -cofibration, as required.

For statement (a), it will now be enough to apply Corollary 2.6 and to show that for a trivial cofibration $f : X \rightarrow Y$, we get a couniversal weak equivalence $f\square Z : X\square Z \rightarrow Y\square Z$ in $[\mathbb{C}, \mathcal{E}]$. For this, observe that like before $f\square Z$ is a pointwise retract of $(i_!i^*f)\square(i_!i^*Z)$. The latter evaluated at $c \in \mathbb{C}_0$ is given by coproduct (7) above. Since the monoid axiom holds in \mathcal{E} , each component of this coproduct is as well a couniversal weak equivalence as well a \otimes -cofibration. Writing this coproduct as a transfinite composition of pushouts of its components shows (in virtue of Lemma 2.3) that the coproduct itself is a couniversal weak equivalence. Since couniversal weak equivalences in $[\mathbb{C}, \mathcal{E}]$ are pointwise couniversal weak equivalences and since they are closed under retract, $f\square Z$ is indeed a couniversal weak equivalence.

For statement (b), observe first that since colimits in $[\mathbb{C}, \mathcal{E}]$ are computed pointwise, and since the weak equivalences in $[\mathbb{C}, \mathcal{E}]$ are the pointwise weak equivalences, the h -cofibrations in $[\mathbb{C}, \mathcal{E}]$ are precisely the pointwise h -cofibrations. Therefore, a similar argument as above (based on Proposition 2.5) yields (b). Statement (c) follows easily from Lemma 1.4ii. \square

Remark 4.2. This theorem recovers and strengthens Theorem 4.4 and Corollary 4.8 of Dundas-Ostvaer-Roendigs [18]. We do not talk about right properness here but right properness is preserved under any transfer. If \mathcal{E} possesses a sufficiently nice system of *spheres* (with *symmetries*) then the formalism of [18] enables one to define *(symmetric) spectra* in \mathcal{E} , as \mathbb{C} -enriched functors on a certain \mathcal{E} -enriched category \mathbb{C} which satisfies assumption (ii) above. Therefore, there exists a levelwise projective model structure on *(symmetric) spectra* in any compactly generated monoidal model category \mathcal{E} with nice system of spheres (with symmetries). This projective model structure is thus h -monoidal whenever \mathcal{E} is. In this special case, h -monoidality could also be derived from Proposition 1.9, since there is a suitable *injective* model structure on *(symmetric) spectra* witnessing the fact that \mathbb{C} is a (generalized) \mathcal{E} -enriched Reedy category.

Remark 4.3. Any one-object \mathcal{E} -enriched symmetric monoidal category \mathbb{C} can be viewed as a commutative monoid in \mathcal{E} and vice-versa. In this case, the diagram category $[\mathbb{C}, \mathcal{E}]$ (equipped with the Day convolution product) may be identified with the category of \mathbb{C} -modules (equipped with the usual tensor product of \mathbb{C} -modules). Theorem 4.1 for this special case recovers one of the results of Schwede-Shiely [46].

Part 2. Model structure for algebras over polynomial monads

In this second part we study algebras over polynomial monads and show that the techniques of Part 1 are applicable to them. Polynomial monads are intermediate between non-symmetric and symmetric colored operads. They have remarkable properties which among others allow a thorough combinatorial analysis of free algebra extensions. Beside the prototypical example of the free monoid monad, most of the currently used notions of operads are expressible as algebras over polynomial monads. Part 3 treats these examples in more detail. The reader may wish to go forth and back between Part 2 and 3 so as to have concrete examples at hand.

5. CARTESIAN MONADS AND THEIR INTERNAL ALGEBRA CLASSIFIERS

In this section we recall the theory of *internal algebra classifiers* of the first author [5] including its recent development in [9]. This tool is fundamental for us because it will enable us (in Section 7) to replace free algebra extensions by left Kan extensions which are much easier to analyse. We formulate the theory for general cartesian monads, though later on we shall only apply it to polynomial monads.

5.1. Cartesian monads. Recall that a natural transformation between two functors is called *cartesian* if all naturality squares are pullbacks. A *monad* T on a category with pullbacks is called *cartesian* if T preserves pullbacks and both, the multiplication and the unit of the monad T , are cartesian natural transformations.

Let T be a cartesian monad on a finitely complete category \mathbb{C} . We denote $\text{Cat}(\mathbb{C})$ the 2-category of categories in \mathbb{C} . Then T induces a monad on $\text{Cat}(\mathbb{C})$ which is enriched in categories. Such a monad is called a *2-monad*. Strict algebras for this 2-monad are called *categorical T -algebras* in \mathbb{C} . As usual, categorical T -algebras can either be considered as categories in T -algebras, or as T -algebras in categories. They form a 2-category with respect to strict morphisms of categorical T -algebra and T -natural transformations.

Definition 5.2. *Let A be a categorical T -algebra in \mathbb{C} .*

An internal T -algebra X in A is a lax morphism of categorical T -algebras $X : 1 \rightarrow A$, where 1 is the terminal categorical T -algebra in \mathbb{C} .

The internal T -algebras in A form a category $\text{Int}_T(A)$ and this correspondence defines a 2-functor:

$$\text{Int}_T : \text{Alg}_T(\text{Cat}(\mathbb{C})) \rightarrow \text{Cat}.$$

Theorem 5.3 ([5]). *The 2-functor Int_T is representable by a categorical T -algebra \mathbf{T}^T . The underlying categorical object of \mathbf{T}^T is the 2-truncated simplicial object*

$$(8) \quad T(1) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} T^2(1) \begin{array}{c} \xleftarrow{\quad} \\ \xrightarrow{\quad} \end{array} T^3(1)$$

of the simplicial bar-resolution $B(T, T, 1)_\bullet$ of the terminal categorical T -algebra 1 .

This categorical T -algebra \mathbf{T}^T will be called the *internal algebra classifier* of T because of its universal property.

Remark 5.4. The free-forgetful adjunction $U_T : \text{Alg}_T \rightleftharpoons \mathbb{C} : F_T$ induces a canonical simplicial bar-resolution $B(T, T, 1)_\bullet \rightarrow 1$ of the terminal T -algebra 1 . Explicitly, $B(T, T, 1)_n = T^{n+1}(1)$, $n \geq 0$ with the usual simplicial operators, induced by

multiplication and unit of T . Since T is a *cartesian*, this simplicial bar-resolution is completely determined by its 2-skeleton and the so-called *Segal maps*

$$B(T, T, 1)_n \longrightarrow \overbrace{B(T, T, 1)_1 \times_{B(T, T, 1)_0} \cdots \times_{B(T, T, 1)_0} B(T, T, 1)_1}^n$$

since the cartesianness of T readily implies that all Segal maps are isomorphisms. In other words, for a cartesian monad T , the bar resolution $B(T, T, 1)_\bullet$ is the simplicial nerve of an essentially unique category \mathbf{T}^T in T -algebras.

5.5. Monad morphisms. Let S (resp. T) be a finitary monad on a cocomplete category \mathbb{D} (resp. \mathbb{C}). For any functor $d : \mathbb{C} \rightarrow \mathbb{D}$ with left adjoint $c : \mathbb{D} \rightarrow \mathbb{C}$ the following three conditions are equivalent:

- (1) There exists a functor $d' : \text{Alg}_T \rightarrow \text{Alg}_S$ such that $U_S d' = d U_T$;
- (2) There exists a natural transformation $\Psi : Sd \rightarrow dT$ compatible with the multiplication and unit of S and T ;
- (3) There exists a morphism of monads $\Phi : S \rightarrow dTc$.

The equivalence between (1) and (2) is classical and does not require the existence of a left adjoint c . For the equivalence between (2) and (3), use unit $\eta : id_{\mathbb{D}} \rightarrow d c$ and counit $\epsilon : c d \rightarrow id_{\mathbb{C}}$ of the adjunction to define $\Phi = \Psi c \circ S \eta$, resp. $\Psi = d T \epsilon \circ \Phi d$. A 2-categorical diagram chase shows that these two assignments are mutually inverse.

If these conditions are satisfied then by the adjoint lifting theorem the functor d' has a left adjoint c' such that the following square of adjoint functors commutes:

$$(9) \quad \begin{array}{ccc} \text{Alg}_S & \xrightleftharpoons[c']{d'} & \text{Alg}_T \\ \begin{array}{c} U_S \downarrow \\ \uparrow F_S \end{array} & & \begin{array}{c} U_T \downarrow \\ \uparrow F_T \end{array} \\ \mathbb{D} & \xrightleftharpoons[c]{d} & \mathbb{C} \end{array}$$

The natural transformation $\Phi : S \rightarrow dTc$ yields (by twofold application of adjunction) a natural transformation $F_T c S \rightarrow F_T c$, which after application of U_T gives a natural transformation

$$\theta : T c S \rightarrow T c$$

inducing the structure of a *right S -module* on the composite functor $T c : \mathbb{D} \rightarrow \mathbb{C}$.

Proposition 5.6. *In the situation above, assume that \mathbb{C} and \mathbb{D} have pullbacks, that T and S are cartesian monads, and that $c \dashv d$ is a cartesian adjunction (i.e. unit and counit are cartesian natural transformations and c preserves pullbacks). Then the following two conditions are equivalent:*

- (i) *the natural transformation $\Phi : S \rightarrow dTc$ is cartesian;*
- (ii) *the natural transformation $\theta : T c S \rightarrow T c$ is cartesian.*

Proof. We leave the proof as an exercise for the reader. □

Definition 5.7. *We will say that Φ is a cartesian morphism from S to T if the equivalent conditions of Proposition 5.6 are satisfied.*

Definition 5.8. Let Φ be a cartesian morphism from S to T . Let A be a categorical T -algebra. An internal S -algebra in A is a lax morphism of categorical S -algebras $X : 1 \rightarrow d'(A)$. There is a 2-functor

$$\text{Int}_S : \text{Alg}_T(\text{Cat}(\mathbb{C})) \rightarrow \text{Cat}$$

which associates to A the category of internal S -algebras in A .

Theorem 5.9 ([5]). The 2-functor Int_S is representable by a categorical T -algebra \mathbf{T}^S . The underlying categorical object of \mathbf{T}^S is the 2-truncated simplicial object

$$(10) \quad Tc(1) \quad \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \quad TcS(1) \quad \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \quad TcS^2(1)$$

of the two-sided bar-construction $B(Tc, S, 1)_\bullet$ where 1 is a terminal categorical S -algebra. In particular, source and target maps are given by $\theta(1)$ and $Tc(!)$ respectively, where $!$ is the unique map from $S(1)$ to 1 .

This categorical T -algebra will be called the *internal S -algebra classifier* of T because of its universal property.

Remark 5.10. Since Φ is cartesian, Proposition 5.6 shows that the right S -module structure of Tc is cartesian as well, so that $B(Tc, S, 1)_\bullet$ is again the simplicial nerve of an essentially unique category \mathbf{T}^S in T -algebras.

5.11. Internal left Kan extensions. For each categorical T -algebra A , the monad morphism $\Phi : S \rightarrow dTc$ induces a functor

$$\delta_A^\Phi : \text{Int}_T(A) \rightarrow \text{Int}_S(A),$$

which should be understood as an internalization of the functor $d' : \text{Alg}_T \rightarrow \text{Alg}_S$. In good cases, the functor δ_A^Φ admits a left adjoint

$$\gamma_A^\Phi : \text{Int}_S(A) \rightarrow \text{Int}_T(A).$$

It is one of the crucial observations of [5] that this left adjoint γ_A^Φ (if it exists) can be computed as a left Kan extension. Indeed, δ_A^Φ may be identified with restriction along a certain functor of categorical T -algebras

$$\mathbf{T}^\Phi : \mathbf{T}^S \rightarrow \mathbf{T}^T$$

in the following way: represent an internal T -algebra X in A by $\tilde{X} : \mathbf{T}^T \rightarrow A$. Then the internal S -algebra $\delta_A^\Phi(X)$ is represented by the composite functor

$$\mathbf{T}^S \xrightarrow{\mathbf{T}^\Phi} \mathbf{T}^T \xrightarrow{\tilde{X}} A.$$

On objects, the functor $\mathbf{T}^\Phi : \mathbf{T}^S \rightarrow \mathbf{T}^T$ is given by $T(!)$, where $! : c(1) \rightarrow 1$, on morphisms it is the composite functor $TcS(1) \rightarrow T^2c(1) \rightarrow T^2(1)$.

Observe that the identity functor $\mathbf{T}^T \rightarrow \mathbf{T}^T$ represents a “universal” internal T -algebra $1 \rightarrow \mathbf{T}^T$. Each internal T -algebra $X : 1 \rightarrow A$ can be recovered from its representation $\tilde{X} : \mathbf{T}^T \rightarrow A$ through precomposition with the universal one $1 \rightarrow \mathbf{T}^T$.

Proposition 5.12 ([5]). Let A be a categorical T -algebra and let Y be an internal S -algebra in A , represented by $\tilde{Y} : \mathbf{T}^S \rightarrow A$. If the left adjoint $\gamma_A^\Phi : \text{Int}_S(A) \rightarrow \text{Int}_T(A)$ exists then $\gamma_A^\Phi(Y)$ can be computed as the composite lax morphism

$$1 \longrightarrow \mathbf{T}^T \xrightarrow{(\mathbf{T}^\Phi)_!(\tilde{Y})} A,$$

where $1 \rightarrow \mathbf{T}^T$ is the universal internal T -algebra and $(\mathbf{T}^\sharp)_!$ is left Kan extension along \mathbf{T}^\sharp in the 2-category $\text{Alg}_T(\text{Cat}(\mathbb{C}))$.

Definition 5.13 ([9]). Let A be a categorical T -algebra with its structure morphism $k : T(A) \rightarrow A$ and $\xi : B \rightarrow C$ be a functor of categorical T -algebras. The algebra A is called *cocomplete with respect to ξ* if for any $X : B \rightarrow A$ the following pointwise left Kan extension in $\text{Cat}(\mathbb{C})$ exists

$$\begin{array}{ccc} B & \xrightarrow{X} & A \\ \xi \downarrow & \phi \downarrow & \nearrow L \\ C & & \end{array}$$

and the induced diagram

$$\begin{array}{ccccc} T(B) & \xrightarrow{T(X)} & T(A) & \xrightarrow{k} & A \\ T(\xi) \downarrow & T(\phi) \downarrow & \nearrow T(L) & & \\ T(C) & & & & \end{array}$$

exhibits $k \circ T(L)$ as the pointwise left Kan extension of $k \circ T(X)$ in $\text{Cat}(\mathbb{C})$.

Theorem 5.14 ([9]). Let A be a categorical T -algebra which is cocomplete with respect to $\mathbf{T}^\sharp : \mathbf{T}^S \rightarrow \mathbf{T}^T$ and let Y be an internal S -algebra in A . Then the pointwise left Kan extension of \tilde{Y} along \mathbf{T}^\sharp in $\text{Alg}_T(\text{Cat}(\mathbb{C}))$ exists and its underlying functor is the pointwise left Kan extension of $U_T(\tilde{Y})$ along $U_T(\mathbf{T}^\sharp)$ in $\text{Cat}(\mathbb{C})$. In particular,

$$U_T(\gamma_A^\Phi(Y)) = \text{colim}_{\mathbf{T}^S} U_T(\tilde{Y}) \quad \text{in } \text{Cat}(\mathbb{C}).$$

A useful generalisation is the following relative version. Let

$$\begin{array}{ccc} R & \xrightarrow{\Phi} & S \\ & \searrow & \swarrow \\ & T & \end{array}$$

be a commutative triangle of cartesian maps between cartesian monads (cf. Definition 5.7). As above it generates a morphism of categorical T -algebras $\mathbf{T}^\sharp : \mathbf{T}^R \rightarrow \mathbf{T}^S$.

Theorem 5.15 ([9]). Let A be a categorical T -algebra which is cocomplete with respect to $\mathbf{T}^\sharp : \mathbf{T}^R \rightarrow \mathbf{T}^S$ and let Y be an internal R -algebra in A . Then the pointwise left Kan extension of \tilde{Y} along \mathbf{T}^\sharp in $\text{Alg}_T(\text{Cat}(\mathbb{C}))$ exists and its underlying functor is the pointwise left Kan extension of $U_T(\tilde{Y})$ along $U_T(\mathbf{T}^\sharp)$ in $\text{Cat}(\mathbb{C})$. In other words, the following diagram of adjoint functors commutes:

$$\begin{array}{ccc}
\mathrm{Int}_R(A) & \xrightleftharpoons[\gamma_A^\phi]{\delta_A^\phi} & \mathrm{Int}_S(A) \\
\downarrow U_T(-) & & \downarrow U_T(-) \\
[U_T(\mathbf{T}^R), U_T(A)] & \xrightleftharpoons[(U_T \mathbf{T}^\Phi)_!]{(U_T \mathbf{T}^\Phi)^*} & [U_T(\mathbf{T}^S), U_T(A)]
\end{array}$$

6. POLYNOMIAL MONADS

In this section we recall the definition of a polynomial monad in sets and of its associated coloured symmetric operad. For a nice and instructive account of polynomial functors we recommend Kock's article [31] from which we shall borrow the idea of representing colored bouquets as certain special polynomials. For a general treatment of polynomial monads in locally cartesian closed categories we refer the reader to Gambino-Kock [20].

6.1. Polynomial functors. For any set I we denote Set/I the comma category over I . Objects of Set/I are mappings $\pi : X \rightarrow I$, and morphisms of Set/I are commuting triangles over I . For each $i \in I$, the preimage $\pi^{-1}(i)$ will be called the *fiber* of π over i . The mapping π is completely determined by its fibers, and hence the category Set/I may be identified with the category of I -indexed families of sets $(X_i)_{i \in I}$. This will be our favourite notation for the objects of Set/I .

Definition 6.2. A polynomial $P = (s, p, t)$ is a diagram in sets of the form

$$J \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

A polynomial is of finite type if all fibers of the middle arrow p are finite.

Each polynomial P generates a functor between comma categories

$$\underline{P} : \mathrm{Set}/J \rightarrow \mathrm{Set}/I$$

which is defined as the composite functor

$$\mathrm{Set}/J \xrightarrow{s^*} \mathrm{Set}/E \xrightarrow{p_*} \mathrm{Set}/B \xrightarrow{t_!} \mathrm{Set}/I$$

where s^* is the pullback functor,

$$s^*(X)_e = X_{s(e)},$$

p_* is right adjoint to p^* ,

$$p_*(X)_b = \prod_{e \in p^{-1}(b)} X_e,$$

and $t_!$ is left adjoint to t^* ,

$$t_!(X)_i = \prod_{b \in t^{-1}(i)} X_b.$$

Any functor \underline{P} generated by a polynomial P is called a *polynomial functor*. In particular, polynomial functors preserve pullbacks. There are several ways to characterize polynomial functors, cf. [31, 20]. One can check that polynomial functors compose [31]. The composite functor $\underline{P} \circ \underline{Q}$ is the polynomial functor \underline{PQ} generated by an up to unique isomorphism uniquely determined polynomial PQ . Cartesian

natural transformations of polynomial functors correspond bijectively to commutative diagrams of the form

$$\begin{array}{ccccccc}
 J & \xleftarrow{s'} & E' & \xrightarrow{p'} & B' & \xrightarrow{t'} & I \\
 \downarrow 1_J & & \downarrow & \lrcorner & \downarrow & & \downarrow 1_I \\
 J & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I
 \end{array}$$

in which the horizontal lines are polynomials and the middle square is a pull-back square. This defines a 2-category **Poly** with 0-cells the comma categories Set/I , with 1-cells the polynomial functors, and with 2-cells the cartesian natural transformations. We denote $\mathbf{Poly}(I)$ the category of polynomial endofunctors $\text{Set}/I \rightarrow \text{Set}/I$ and cartesian natural transformations. It is a monoidal category for composition of endofunctors.

Definition 6.3. *A polynomial monad is a monad in the 2-category **Poly**.*

Hence a polynomial monad T over I is a monoid in $(\mathbf{Poly}(I), \circ)$. Each polynomial monad over I generates a cartesian monad on Set/I . The polynomial monad is finitary if and only if the generating polynomial is of finite type.

Finitary polynomial monads over I correspond to certain I -coloured symmetric operads in sets. Kock [31] and Szawiel-Zawadowski [44] made this correspondence explicit and showed that finitary polynomial monads correspond to those coloured symmetric operads for which the symmetry group actions are free.

Definition 6.4 (cf. [31]). *An I -coloured bouquet of arity k is a polynomial*

$$I \xleftarrow{s} \{1, 2, \dots, k\} \xrightarrow{p} \{1\} \xrightarrow{t} I.$$

The latter will be represented by the $(k+1)$ -tuple $(s(1), \dots, s(k); t(1)) \in I^{k+1}$.

The full subcategory of $\mathbf{Poly}(I)$ spanned by I -coloured bouquets will be denoted $\mathbf{Bq}(I)$. The associated nerve functor is denoted

$$\begin{array}{rcl}
 \mathcal{O} : \mathbf{Poly}(I) & \rightarrow & \mathbf{Coll}(I) = \text{Set}^{\mathbf{Bq}(I)^{\text{op}}} \\
 P & \mapsto & \mathcal{O}_P = \text{Hom}_{\mathbf{Poly}(I)}(-, P)
 \end{array}$$

The subcategory $\mathbf{Bq}(I)$ is dense in $\mathbf{Poly}(I)$, i.e. the nerve functor is fully faithful. Moreover, $\mathbf{Bq}(I)$ is a groupoid: the symmetry group of a bouquet of arity k may be identified with a certain subgroup of the symmetry group of $\{1, \dots, k\}$. The essential image of the nerve functor consists of those I -coloured collections in $\mathbf{Coll}(I)$ for which the automorphisms in $\mathbf{Bq}(I)$ act freely, cf. [31, 2.4.10].

There is a substitutional \circ -product on $\mathbf{Coll}(I)$ for which the monoids are precisely the I -coloured symmetric operads in sets, cf. the appendix of [11], where the category of I -coloured bouquets $\mathbf{Bq}(I)$ is denoted $\mathbb{F}^{\leq}(I)$. It can be checked by hand that the nerve functor is a *monoidal* functor

$$\mathcal{O} : (\mathbf{Poly}(I), \circ) \rightarrow (\mathbf{Coll}(I), \circ)$$

and therefore takes polynomial monads over I to I -coloured symmetric operads. It follows from [31, 2.2.12] that this “enhanced” nerve functor induces an equivalence between the category of finitary polynomial monads over I and the category of I -coloured symmetric operads with freely acting symmetry groups, cf. also [44].

From now on we always assume that our polynomial monads are finitary.

Remark 6.5. Let T be a polynomial monad generated by the polynomial

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I$$

We can give the following elementary description of coloured symmetric operad \mathcal{O}_T associated to T . Each element $b \in B$ comes equipped with a target $t(b) = i \in I$, and a fiber $p^{-1}(b) \subset E$. The elements $e \in p^{-1}(b)$ of the fiber have sources $s(e) \in I$.

An *operation* of T is a pair (b, σ) consisting of an element $b \in B$ and a bijection $\sigma : \{1, 2, \dots, k\} \rightarrow p^{-1}(b)$. We shall refer to σ as a *linear ordering*. Such an operation (b, σ) belongs to $\mathcal{O}_T(i_1, \dots, i_k; i)$ precisely when $t(b) = i$ and $(s(\sigma(1)), s(\sigma(2)), \dots, s(\sigma(k))) = (i_1, \dots, i_k)$. We shall write $s(b, \sigma) = (i_1, \dots, i_k)$ and $t(b, \sigma) = i$ and call them source and target of the operation (b, σ) .

A *composite operation* is a list of operations $((b, \sigma); (b_1, \sigma_1), (b_2, \sigma_2), \dots, (b_k, \sigma_k))$ such that $t(b_i) = s(\sigma(i))$. With these notations the multiplication of the monad T associates to each composite T -operation $((b, \sigma); (b_1, \sigma_1), (b_2, \sigma_2), \dots, (b_k, \sigma_k))$ a single T -operation $(b(b_1, \dots, b_k), \sigma(\sigma_1, \dots, \sigma_k))$ with same target as b and with source-list $(s(b_1, \sigma_1), s(b_2, \sigma_2), \dots, s(b_k, \sigma_k))$ linearly ordered by $\sigma(\sigma_1, \dots, \sigma_k)$ in the obvious way. This multiplication satisfies the usual associativity, unitarity and equivariance constraints of an I -coloured symmetric operad.

6.6. Algebras over polynomial monads. The category of T -algebras Alg_T for a polynomial monad T on Set/I coincides with the category of \mathcal{O}_T -algebras of the associated coloured symmetric operad \mathcal{O}_T . Explicitly a T -algebra in sets is given by an I -indexed family of sets $(A_i)_{i \in I}$ together with structural maps

$$m_{(b, \sigma)} : A_{s(\sigma(1))} \times \dots \times A_{s(\sigma(k))} \rightarrow A_{t(b)}$$

for each operation (b, σ) of T . These structure maps satisfy the usual associativity, unitarity and equivariance conditions of an algebra over a coloured symmetric operad.

Given a cocomplete symmetric monoidal category $(\mathcal{E}, \otimes, e)$, the strong symmetric monoidal functor

$$\text{Set} \rightarrow \mathcal{E} : X \mapsto \coprod_X e$$

takes the coloured symmetric operad \mathcal{O}_T to a coloured symmetric operad in \mathcal{E} and thus defines a category $\text{Alg}_T(\mathcal{E})$ of T -algebras in \mathcal{E} . Explicitly, a T -algebra A in \mathcal{E} is an I -indexed family $(A_i)_{i \in I}$ of objects of \mathcal{E} together with structural maps

$$m_{(b, \sigma)} : A_{s(\sigma(1))} \otimes \dots \otimes A_{s(\sigma(k))} \rightarrow A_{t(b)}$$

for each operation (b, σ) of T , subject to the same associativity, unitarity and equivariance conditions as above.

6.7. Internal T -algebras in cocomplete symmetric monoidal categories.

Since each polynomial monad T is cartesian it generates a 2-monad on the category Cat/I where I is considered as a discrete category. The category of strict algebras of this 2-monad is by definition the category $\text{Alg}_T(\text{Cat})$ of categorical T -algebras. There is also a 2-category of *pseudo- T -algebras* associated to the 2-monad T . By a strictification theorem of [5], any pseudo- T -algebra is equivalent to a strict T -algebra. We shall tacitly apply this strictification whenever necessary.

It is not difficult to see that a categorical T -algebra $(A_i)_{i \in I}$ is cocomplete (in the sense of Definition 5.13) with respect to arbitrary morphisms between small

categorical T -algebras if and only if A_i is a cocomplete category for all $i \in I$ and the structure maps $m_{(b,\sigma)} : A_{s(\sigma(1))} \times \dots \times A_{s(\sigma(k))} \rightarrow A_{t(b)}$ preserve colimits in each variable. From now on such categorical T -algebras will be called *cocomplete*.

Let A be a categorical T -algebra. Then an internal T -algebra in A can be explicitly given by a collection of objects $a_i \in A_i$ together with a morphism

$$\mu_{(b,\sigma)} : m_{(b,\sigma)}(a_{s(\sigma(1))}, \dots, a_{s(\sigma(k))}) \rightarrow a_{t(b)},$$

for each operation (b, σ) , which satisfies obvious associativity, unitarity and equivariance conditions. Here, $m_{(b,\sigma)}$ is the structure functor of A .

To any symmetric monoidal category $(\mathcal{E}, \otimes, e)$ we associate the categorical pseudo- T -algebra \mathcal{E}_T^\bullet with constant underlying collection

$$\mathcal{E}_T^\bullet(i) = \mathcal{E}, \quad i \in I.$$

Nullary T -operations act as unit $e : 1 \rightarrow \mathcal{E}$, unary T -operations act as identity and T -operations of arity $n \geq 2$ acts as iterated tensor product \otimes^n . This pseudo- T -algebra \mathcal{E}_T^\bullet is cocomplete if and only if \mathcal{E} is cocomplete as a symmetric monoidal category, i.e. if and only if \mathcal{E} is cocomplete as a category, and moreover the tensor of \mathcal{E} commutes with colimits in both variables.

The assignment $\mathcal{E} \mapsto \mathcal{E}_T^\bullet$ is the right adjoint part of an adjunction between categorical T -algebras and symmetric monoidal categories. This adjunction is induced by a map $T \rightarrow P$ of polynomial monads in \mathbf{Cat} where P denotes the monad for symmetric monoidal categories. The existence of this adjunction provides a conceptual reason for the existence of an enrichment over \mathcal{E} of the category of T -algebras. The interested reader may find more details in [9]. We will not pursue this point of view any further here. However, it will be essential for us to represent T -algebras in \mathcal{E} as internal T -algebras in \mathcal{E}_T^\bullet , based on the following proposition.

Proposition 6.8 ([9]). *The category of T -algebras in \mathcal{E} is isomorphic to the category of internal T -algebras in \mathcal{E}_T^\bullet .*

6.9. Representing T -algebras as functors $\mathbf{T}^\mathbf{T} \rightarrow \mathcal{E}_T^\bullet$. –

Let us begin by describing the internal T -algebra classifier $\mathbf{T}^\mathbf{T}$. By definition, the objects of $\mathbf{T}^\mathbf{T}$ are the elements of the free T -algebra $T(1)$ on 1. These elements correspond bijectively to the elements of B . A morphism from b' to b is given by an element of $T^2(1)$ with correct source and target. Such an element corresponds to a $(k+1)$ -tuple $(b; b_1, \dots, b_k) \in B^{k+1}$ (for varying $k \geq 1$) such that for linear orderings $\sigma, \sigma_1, \dots, \sigma_k$, the $(k+1)$ -tuple $((b, \sigma); (b_1, \sigma_1), \dots, (b_k, \sigma_k))$ is a composite T -operation in the sense of Remark 6.5, and such that $b' = b(b_1, \dots, b_k)$.

The unit of the polynomial monad T defines an I -indexed collection $(1_i)_{i \in I}$ of special elements $1_i \in B$. The latter induce two families of morphisms in $\mathbf{T}^\mathbf{T}$, namely the identities

$$b \xrightarrow{(b; 1_{i_1}, \dots, 1_{i_k})} b$$

and the morphisms

$$b \xrightarrow{(1_i; b)} 1_i.$$

Now, let $(X_i)_{i \in I}$ be the I -collection underlying a T -algebra in \mathcal{E} . Then the associated map of categorical T -algebras

$$\tilde{X} : \mathbf{T}^\mathbf{T} \rightarrow \mathcal{E}_T^\bullet$$

is constructed as follows. We must have $\tilde{X}(1_i) = X_i$ and $\tilde{X}(b) = X_{i_1} \otimes \cdots \otimes X_{i_k}$ where (i_1, \dots, i_k) is the source-list of the fiber $p^{-1}(b)$ for a fixed linear ordering σ . Then the map $\tilde{X}(b \rightarrow 1_i)$ in \mathcal{E} represents the action $m_{(b,\sigma)} : X_{i_1} \otimes \cdots \otimes X_{i_k} \rightarrow X_i$, and the functoriality of \tilde{X} amounts precisely to the equivariance, associativity and unitarity constraints of the T -action on $(X_i)_{i \in I}$.

At this point, we should mention two examples which have been decisive for the elaboration of the whole theory. If T is the free monoid monad (see Section 10) then \mathbf{T}^T is isomorphic to the augmented simplex category Δ_+ , and we recover the well-known fact that monoids in \mathcal{E} correspond to strict monoidal functors $\Delta_+ \rightarrow \mathcal{E}$. If T is the free symmetric operad monad (see Section 12) then \mathbf{T}^T is isomorphic to a category of labelled rooted planar trees \mathbf{RT}^{RTr} which goes back to Ginzburg-Kapranov, and which again has the characteristic property that symmetric operads in \mathcal{E} correspond to certain functors $\mathbf{RT}^{\text{RTr}} \rightarrow \mathcal{E}$.

6.10. Cartesian morphisms of polynomial monads. We need a more general notion of map between polynomials which includes the possibility of base-change. Let S be a polynomial monad generated by a polynomial

$$J \xleftarrow{s'} E' \xrightarrow{p'} B' \xrightarrow{t'} J.$$

A cartesian morphism $\Phi = (\delta, \psi, \phi)$ from S to T is a commutative diagram in sets

$$\begin{array}{ccccccc} J & \xleftarrow{s'} & E' & \xrightarrow{p'} & B' & \xrightarrow{t'} & J \\ \delta \downarrow & & \downarrow \psi & \lrcorner & \downarrow \phi & & \downarrow \delta \\ I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \end{array}$$

in which the middle square is a pullback, and the horizontal lines generate the polynomial monads S and T . The mapping $\delta : J \rightarrow I$ induces a cartesian adjunction $c \dashv d$ where $d : \text{Set}/I \rightarrow \text{Set}/J$ is the pullback functor and $c : \text{Set}/J \rightarrow \text{Set}/I$ its left adjoint. Then the equivalent conditions of Proposition 5.6 are fulfilled and Φ generates a cartesian morphism (in the sense of Definition 5.7) from the polynomial monad S to the polynomial monad T .

For a symmetric monoidal category \mathcal{E} we get a restriction functor

$$\delta_{\mathcal{E}}^{\Phi} : \text{Alg}_T(\mathcal{E}) \rightarrow \text{Alg}_S(\mathcal{E}).$$

Observe that $d'(\mathcal{E}_T^{\bullet}) = \mathcal{E}_S^{\bullet}$ so that an internal S -algebra in \mathcal{E}_T^{\bullet} is the same as an ordinary S -algebra in \mathcal{E} . Therefore, the restriction functor $\delta_{\mathcal{E}}^{\Phi}$ is induced by a functor $\mathbf{T}^{\Phi} : \mathbf{T}^S \rightarrow \mathbf{T}^T$ on the level of internal algebra classifiers, and its left adjoint

$$\gamma_{\mathcal{E}}^{\Phi} : \text{Alg}_S(\mathcal{E}) \rightarrow \text{Alg}_T(\mathcal{E})$$

can be calculated as a colimit over \mathbf{T}^S . To carry out such a program we need a description of the internal S -algebra classifier \mathbf{T}^S and of the canonical functor $\mathbf{T}^{\Phi} : \mathbf{T}^S \rightarrow \mathbf{T}^T$ in terms of the given map Φ between the generating polynomials.

6.11. The category \mathbf{T}^S . By definition, the objects of \mathbf{T}^S are the elements of $Tc(1)$. These elements can be understood as J -coloured T -operations. In order to distinguish them from the objects of \mathbf{T}^T we shall denote them by bold letters. A J -coloured T -operation \mathbf{b} is given by an element $b \in B$ together with a chosen colour

$j \in \delta^{-1}(s(e))$ for each element $e \in p^{-1}(b)$ of the fibre of b . On the object level, the canonical functor $\mathbf{T}^\sharp : \mathbf{T}^S \rightarrow \mathbf{T}^T$ is just forgetting the J -colouring. So, an object \mathbf{b} of \mathbf{T}^S is determined by its image $\mathbf{T}^\sharp(\mathbf{b})$ together with a compatible J -colouring.

The component $\theta(1)$ of the right S -action $\theta : TcS \rightarrow Tc$ is defined as follows. Note first that the elements of $TcS(1)$ can be understood as J -coloured T -operations \mathbf{b} together with a compatible family of S -operations $d_1, \dots, d_k \in B'$ in the sense that $(t'(d_1), \dots, t'(d_k))$ coincides with the J -colouring of \mathbf{b} . Then

$$\mathbf{T}^\sharp(\theta(\mathbf{b}; d_1, \dots, d_k)) = \mathbf{T}^\sharp(\mathbf{b})(\phi(d_1), \dots, \phi(d_k)),$$

and the J -colouring of $\theta(\mathbf{b}; d_1, \dots, d_k)$ is inherited from (the sources of) the fibres of d_1, \dots, d_k in an obvious way.

A morphism $\mathbf{b}' \rightarrow \mathbf{b}$ in \mathbf{T}^S is given by an element $(\mathbf{b}; d_1, \dots, d_k)$ of $TcS(1)$ such that $\mathbf{b}' = \theta(\mathbf{b}; d_1, \dots, d_k)$. The effect of \mathbf{T}^\sharp on a morphism $\mathbf{b}' \rightarrow \mathbf{b}$ is obvious: if $\mathbf{b}' = \theta(\mathbf{b}; d_1, \dots, d_k)$ then the identity $\mathbf{T}^\sharp(\mathbf{b}') = \mathbf{T}^\sharp(\mathbf{b})(\phi(d_1), \dots, \phi(d_k))$ represents a morphism $\mathbf{T}^\sharp(\mathbf{b}') \rightarrow \mathbf{T}^\sharp(\mathbf{b})$ in \mathbf{T}^T .

6.12. Representing S -algebras as functors $\mathbf{T}^S \rightarrow \mathcal{E}_T^\bullet$. –

Let X be an S -algebra in \mathcal{E} . By the universal property of \mathbf{T}^S , the corresponding internal S -algebra in \mathcal{E}_T^\bullet is represented by a functor $\tilde{X} : \mathbf{T}^S \rightarrow \mathcal{E}_T^\bullet$. The latter can be described as follows: Let \mathbf{b} be an object of \mathbf{T}^S . Then

$$(11) \quad \tilde{X}(\mathbf{b}) = X_{j_1} \otimes \dots \otimes X_{j_k}$$

where (j_1, \dots, j_k) is the J -colouring of \mathbf{b} . The morphism $\tilde{X}(\mathbf{b}' \rightarrow \mathbf{b})$ in \mathcal{E} is defined by the action of the S -operations d_1, \dots, d_k on X , where $\mathbf{b}' = \theta(\mathbf{b}; d_1, \dots, d_k)$.

Theorem 6.13. *Let \mathcal{E} be a cocomplete symmetric monoidal category and let Φ be a cartesian morphism from the polynomial monad S to the polynomial monad T .*

Then the restriction functor $\delta_{\mathcal{E}}^\Phi$ admits a left adjoint $\gamma_{\mathcal{E}}^\Phi : \text{Alg}_S(\mathcal{E}) \rightarrow \text{Alg}_T(\mathcal{E})$. For any S -algebra X in \mathcal{E} , the underlying I -collection of the T -algebra $\gamma_{\mathcal{E}}^\Phi(X)$ can be calculated as the following colimit:

$$(12) \quad \gamma_{\mathcal{E}}^\Phi(X)_i = \text{colim}_{\mathbf{b} \in \mathbf{T}^S, t(\mathbf{b})=i} \tilde{X}(\mathbf{b})$$

Proof. This follows immediately from Theorem 5.14. \square

For any polynomial monad T the unit of the monad is a cartesian map of polynomial monads. It is easy to see that the category \mathbf{T}^{Id} has the same objects as \mathbf{T}^T but only identity morphisms. Moreover, I -indexed collections in \mathcal{E} can be identified with algebras over the identity monad, and hence with functors $\tilde{X} : \mathbf{T}^{\text{Id}} \rightarrow \mathcal{E}_T^\bullet$. This observation allows us to calculate the iterations of the monad T in terms of morphisms in \mathbf{T}^T . We have

Corollary 6.14. *Let $(X_i)_{i \in I}$ be a collection. Then*

$$T^k(X)_{i_0} = \coprod_{i_0 \xrightarrow{f_1} b_1 \dots b_{k-1} \xrightarrow{f_k} b_k} \tilde{X}(b_k),$$

where f_1, \dots, f_{k-1} are composable morphisms in \mathbf{T}^T .

This formula will be useful for us later.

7. FREE ALGEBRA EXTENSIONS

We are going to apply the theory above to explicit calculations of some pushouts of algebras of cartesian monads. More specifically we will be interested in calculation of pushouts along free maps of algebras and, as a special case, of coproducts of algebra and a free algebra.

Let C be a category and let (T, μ, ϵ) be a monad on C . Let $U : Alg_T \rightarrow C$ be forgetful functor and $F : C \rightarrow Alg_T$ be its left adjoint.

Let $\mathcal{P}_{f,g}$ be the category whose objects are quintuples (X, K, L, g, f) , where X is a T -algebra, K, L are objects in C , $g : K \rightarrow U(X)$, $f : K \rightarrow L$ are morphisms in C . There is a forgetful functor

$$\begin{aligned} \mathcal{W} : \mathcal{P}_{f,g} &\rightarrow C \times C \times C, \\ \mathcal{W}(X, K, L, f, g) &= (U(X), K, L). \end{aligned}$$

Lemma 7.1. *Let C be cocomplete and T be a finitary monad. Then:*

- (1) *The functor \mathcal{W} is monadic and the corresponding monad $(T_{f,g}, \nu, \iota)$ is finitary;*
- (2) *Let \mathcal{F} be a left adjoint to \mathcal{W} . There is a commutative square of adjunctions*

$$(13) \quad \begin{array}{ccc} \mathcal{P}_{f,g} & \begin{array}{c} \xleftarrow{\delta_1} \\ \xrightarrow{P} \end{array} & Alg_T \\ \mathcal{W} \downarrow \uparrow \mathcal{F} & & U \downarrow \uparrow F \\ C \times C \times C & \begin{array}{c} \xleftarrow{\delta_0} \\ \xrightarrow{\sqcup} \end{array} & C \end{array}$$

where $\delta_0 : C \rightarrow C \times C \times C$ is the diagonal functor

$$\delta_0(X) = (X, U(X), U(X), id, id)$$

and $\sqcup(X, K, L) = X \sqcup K \sqcup L$ is the coproduct in C ;

- (3) *If C has pullbacks which commute with coproducts and T is cartesian monad then the monad $(T_{f,g}, \nu, \epsilon)$ is cartesian. and the natural transformation*

$$\theta : F(\sqcup(\mathcal{W}(\mathcal{F}(X, K, L)))) \rightarrow F(\sqcup(X, K, L))$$

is cartesian.

Proof. We construct the left adjoint to \mathcal{W} explicitly. Let

$$\mathcal{F} : C \times C \times C \rightarrow \mathcal{P}_{f,g}$$

be a functor which associates to a triple (X, K, L) the quintuple

$$(14) \quad (F(X \sqcup K), K, K \sqcup L, i, j)$$

where i is the composite

$$K \xrightarrow{\epsilon} T(K) = UF(K) \xrightarrow{T(c')} T(X \sqcup K)$$

and $c' : K \rightarrow X \sqcup K$ is the canonical coprojection and j is the canonical coprojection

$$c_2 : K \rightarrow K \sqcup L.$$

The statements (1) and (2) are elementary to check.

In the statement (3) cartesianess of the monad $T_{f,g}$ is also elementary. The statement about the transformation θ follows from the following description of its components.

On the triple (X, K, L) the transformation θ provides a morphism.

$$F(T(X \sqcup K) \sqcup K \sqcup (K \sqcup L)) \rightarrow F(X \sqcup K \sqcup L).$$

But

$$F(T(X \sqcup K) \sqcup K \sqcup (K \sqcup L)) \simeq F(T(X \sqcup K)) \sqcup F(K) \sqcup F(K \sqcup L),$$

and then the transformation θ is equal on the summand $F(K)$ to a canonical coprojection to $F(X \sqcup K \sqcup L) \simeq F(X) \sqcup F(K) \sqcup F(L)$.

Similarly on the summand $F(K \sqcup L)$. Finally, a component of θ on $F(T(X \sqcup K))$ is the composite of a coprojection and a transformation $\psi : FT(X \sqcup K) \rightarrow F(X \sqcup K)$ adjoint to the identity $T(X \sqcup K) \rightarrow UF(X \sqcup K)$. The transformation $U(\psi)$ is just a multiplication in the monad T and thus is cartesian.

□

It is obvious from universal property that the functor P from Lemma 7.1 is given by the pushout in Alg_T :

$$(15) \quad \begin{array}{ccc} F(K) & \xrightarrow{F(f)} & F(L) \\ \phi \downarrow & & \downarrow \\ X & \longrightarrow & P(X, K, L, g, f) \end{array}$$

where ϕ is induced by g by the universal property of the free algebra functor.

This means that we can apply the theory of internal algebras to explicitly compute pushouts of algebras along a free map. We have a canonical T -algebras map $\mathbf{T}^\sharp : \mathbf{T}^{\mathbf{T}_f, g} \rightarrow \mathbf{T}^{\mathbf{T}}$ and by Theorem 5.14 the underlying object of the pushout of internal algebras in a cocomplete (with respect to ζ) categorical T -algebra along a free map can be computed as a certain colimit over $\mathbf{T}^{\mathbf{T}_f, g}$.

Analogously, let \mathcal{P}_f be the category whose objects are quadruples (X, K, L, f) , where X is a T -algebra, K, L are objects in C , $f : K \rightarrow L$ is a morphism in C .

We obtain then a monad T_f and a categorical T -algebra $\mathbf{T}^{\mathbf{T}_f}$ colimit over which computes the coproduct $X \vee F(L)$ of internal T -algebras.

Let \mathcal{P}_g be the category whose objects are quadruples (X, K, L, g) , where X is a T -algebra, K, L are objects in C and $g : K \rightarrow U(X)$ is a morphism. We obtain again a monad T_g and a corresponding categorical algebra $\mathbf{T}^{\mathbf{T}_g}$.

Let $\mathcal{P} + 2$ be the category whose objects are triples (X, K, L) , where X is a T -algebra, K, L are objects in C . We obtain a monad $T + 2$ and a categorical algebra $\mathbf{T}^{\mathbf{T}+2}$ colimit over which computes the coproduct $X \vee F(K) \vee F(L)$. We have canonical commutative diagram of T -algebras functors

$$\begin{array}{ccc}
\mathbf{T}^{T+2} & \longrightarrow & \mathbf{T}^{T_\varepsilon} \\
\downarrow & & \downarrow \\
\mathbf{T}^{T_g} & \longrightarrow & \mathbf{T}^{T_{\varepsilon,g}}
\end{array}$$

which are identities on objects of objects.

Finally, let $\mathcal{P} + 1$ be the category whose objects are couples (X, K) , where X is a T -algebra, K is an objects in \mathcal{C} . So, we obtain a monad $T + 1$ and a categorical algebra \mathbf{T}^{T+1} colimit over which computes the coproduct $X \vee F(K)$. The coproducts of this type will play special role in our theory so we give them a special name.

Definition 7.2. *A semifree coproduct of a T -algebra X and an object K is the coproduct $X \vee F(K)$ in the category of T -algebras.*

7.3. Tame polynomial monads. According to Section 7 we have canonical categories $\mathbf{T}^{T_{\varepsilon,g}}, \mathbf{T}^{T_\varepsilon}, \mathbf{T}^{T+2}$ and \mathbf{T}^{T+1} associated with our monad T . We need to understand better the structure of these categories to be able to compute colimits over $\mathbf{T}^{T_{\varepsilon,g}}$ (i.e. pushouts along free maps of T -algebras) in a nice way. We first show that the monad $T + 1$ is polynomial so one can use the results of Section 6.11 for an explicit presentation of \mathbf{T}^{T+1} .

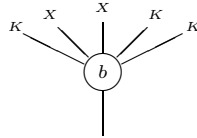
A polynomial, which describes $T + 1$ is given by the following diagram:

$$I \sqcup I \xleftarrow{s \sqcup 1} E \sqcup I \xrightarrow{p \sqcup 1} B \sqcup I \xrightarrow{t \sqcup 1} I \sqcup I.$$

Then we have a cartesian map of monads represented by the diagram

$$\begin{array}{ccccccc}
I \sqcup I & \longleftarrow & E \sqcup I & \xrightarrow{p \sqcup 1} & B \sqcup I & \longrightarrow & I \sqcup I \\
\downarrow \delta & & \downarrow \psi & & \downarrow \phi & & \downarrow \delta \\
I & \longleftarrow & E & \xrightarrow{p} & B & \longrightarrow & I
\end{array}$$

The map δ is the identity on each copy of I , the map ϕ is identity on B and is equal to the unit of the monad on I , the map ψ is defined analogously. So, an object \mathbf{b} of $\mathbf{T}^{T+1}(i)$ is an element $b \in B, t(b) = i$ together with colouring of the elements of $p^{-1}(b)$ by two colours, for which we will use notations X and K . Such an object can be represented by a corolla with coloured edges:



There exists a morphism from $\mathbf{b}' \rightarrow \mathbf{b}$ if there exist operations b_1, \dots, b_k one for each X -coloured edge of \mathbf{b} such that $b' = m(b; 1, \dots, b_1, 1, \dots, b_k, \dots, 1)$ where 1 correspond to K -edges, and an edge of \mathbf{b}' has X -colour if it comes from one of the b_i and K -colour if it comes from 1.

Definition 7.4. *We say that T is a tame polynomial monad if the canonical functor*

$$\mathbf{T}^{T+1} \rightarrow \pi_0(\mathbf{T}^{T+1})$$

has a right adjoint. In other words each connected component of \mathbf{T}^{T+1} has a terminal object.

If T is tame then semifree coproducts admit a functorial polynomial formula similar to the formula (1) from the Introduction. Indeed, if \mathbf{T}^{T+1} has a terminal object in each connected component then the colimit from Theorem 6.13 can be calculated as coproduct of values of \tilde{X} on these terminal objects, that is it can be represented as a coproduct of tensor products of $X_i, K_i, i \in I$. More precisely, there exists a functor

$$P : \text{Set}/I \times \text{Set}/I \rightarrow \text{Set}/I$$

making the following diagram commutative

$$\begin{array}{ccc} \text{Alg}_T \times \text{Set}/I & \xrightarrow{(-) \vee^{F_T} (-)} & \text{Alg}_T \\ \downarrow U_T \times \text{Id} & & \downarrow U_T \\ \text{Set}/I \times \text{Set}/I & \xrightarrow{P} & \text{Set}/I \end{array}$$

and this functor is polynomial. The corresponding polynomial is constructed as follows:

$$I \sqcup I \leftarrow \pi_0(\mathbf{T}^{T+1})^* \rightarrow \pi_0(\mathbf{T}^{T+1}) \rightarrow I.$$

Here we identify $\pi_0(\mathbf{T}^{T+1})$ with the set of objects of \mathbf{T}^{T+1} which are terminal in their connected components. Such an object is represented by a corolla decorated by an element b whose edges have colours X or K . The target of b gives a map $s : \pi_0(\mathbf{T}^{T+1}) \rightarrow I$. The set $\pi_0(\mathbf{T}^{T+1})^*$ is the set of corollas as above with one edge marked. The map $\pi_0(\mathbf{T}^{T+1})^* \rightarrow \pi_0(\mathbf{T}^{T+1})$ simply forgets the marking. The source map $\pi_0(\mathbf{T}^{T+1})^* \rightarrow I \sqcup I$ returns the source of the marked element of $p^{-1}b$ and places it to the first copy of I if the colour of the edge was X and to the second copy of I if the colour was K .

See Sections 10 and 11 for the explicit examples.

7.5. Categories $\mathbf{T}^{T,g}, \mathbf{T}^T, \mathbf{T}^g, \mathbf{T}^{T+2}$ explicitly. Analogously to the case \mathbf{T}^{T+1} we can prove that the monads $T_{f,g}, T_f, T_g$ and $T + 2$ are polynomial and we obtain the following description of the categories $\mathbf{T}^{T,g}, \mathbf{T}^T, \mathbf{T}^g$ and \mathbf{T}^{T+2} . The objects of these categories coincide and consist of corollas labeled by the elements of B whose edges are coloured by one of three colours X, K, L . An edge of such a corolla will be called X -edge, K -edge or L -edge according to its colour.

The morphisms in $\mathbf{T}^{T,g}$ can be described in terms of generators and relations. We have three different classes of generators. First, we have generators coming from T -algebra structure on 1. Such a generator is described similarly to the case of morphisms in \mathbf{T}^{T+1} . Relations come from the relations between operations in T . Let M be the class of morphisms in $\mathbf{T}^{T,g}$ generated by the above generators only.

Next type of generators correspond to the morphism $f : K \rightarrow L$. Such a generator simply replaces a K -edge by an L -edge in the corolla. The class generated by those morphisms is denoted F . Finally, we have generators which correspond to $g : K \rightarrow X$. Again such a morphism replaces a K -edge by an X -edge. The class generated by these morphisms is denoted G . The relations between morphisms from these three classes are obvious from the description.

In \mathbf{T}^T the class G is empty, in \mathbf{T}^g the class F is empty and in \mathbf{T}^{T+2} the classes F, G are empty.

8. ADMISSIBILITY OF TAME POLYNOMIAL MONADS

In this section we assume that T is a tame polynomial monad.

8.1. The final subcategory $\mathbf{t} \subset \mathbf{T}^{\mathbf{T}_{\mathbf{f},\mathbf{g}}}$.

Lemma 8.2. *The categories $\mathbf{T}^{\mathbf{T}_{\mathbf{f}}}, \mathbf{T}^{\mathbf{T}_{\mathbf{g}}}, \mathbf{T}^{\mathbf{T}+2}$ have a terminal objects in each connected component.*

Proof. This property is obvious for $\mathbf{T}^{\mathbf{T}+2}$ under the assumption of tameity. If a is a terminal object in a connected component in $\mathbf{T}^{\mathbf{T}+2}$ then replacing all K -edges by L -edges gives us a terminal object in the connected component of a in $\mathbf{T}^{\mathbf{T}_{\mathbf{f}}}$. Analogously, starting from an object a in $\mathbf{T}^{\mathbf{T}_{\mathbf{g}}}$ and applying morphisms in G we construct a unique morphism in $\mathbf{T}^{\mathbf{T}_{\mathbf{g}}}$ from a to an object without K -edges, that is having only X and L -edges. So, we are in the subcategory of $\mathbf{T}^{\mathbf{T}_{\mathbf{f}}}$ which is isomorphic to $\mathbf{T}^{\mathbf{T}+1}$ and it does have a terminal object. Obviously, this terminal object is also a terminal object in the connected component of a .

□

Let \mathbf{t}_0 be the discrete final subcategory of $\mathbf{T}^{\mathbf{T}+2}$ consisting of chosen terminal objects in each connected component. We can factorize the functor

$$\mathbf{t}_0 \rightarrow \mathbf{T}^{\mathbf{T}+2} \rightarrow \mathbf{T}^{\mathbf{T}_{\mathbf{f},\mathbf{g}}}$$

as composite of bijective on objects functor followed by a fully faithful functor

$$\mathbf{t}_0 \xrightarrow{H} \mathbf{t} \xrightarrow{E} \mathbf{T}^{\mathbf{T}_{\mathbf{f},\mathbf{g}}}.$$

Lemma 8.3. *The functor E is final.*

Proof. By definition for every object of $a \in \mathbf{T}^{\mathbf{T}_{\mathbf{f},\mathbf{g}}}$ there exists an object from $b \in \mathbf{t}$ with a morphism $\phi : a \rightarrow b$ in $\mathbf{T}^{\mathbf{T}+2}$. But $\mathbf{T}^{\mathbf{T}+2}$ is the subcategory of $\mathbf{T}^{\mathbf{T}_{\mathbf{f},\mathbf{g}}}$. So, a/\mathbf{t} is not empty.

We prove that a/\mathbf{t} is connected by induction on number k of occurrence of the K -edges in a .

Let $k = 0$ then all morphisms from a are from M . These morphisms are in $\mathbf{T}^{\mathbf{T}+2}$ and so there is only one object in a/\mathbf{t} .

Assume now that we already proved that the category a'/\mathbf{t} is connected for all a' which have the number of K -edges less or equal to k . Let a has $k + 1$ K -edges.

Let $\lambda_b : a \rightarrow b$ and $\lambda_c : a \rightarrow c$ be two objects of a/\mathbf{t} , and let $\lambda_b = \zeta_1 \cdot \zeta_2 \cdot \dots \cdot \zeta_n$ and $\lambda_c = \xi_1 \cdot \xi_2 \cdot \dots \cdot \xi_m$, where ζ_i, ξ_j are generators in $\mathbf{T}^{\mathbf{T}_{\mathbf{f},\mathbf{g}}}$. The generators in $\mathbf{T}^{\mathbf{T}_{\mathbf{f},\mathbf{g}}}$ belong to the three classes of morphisms M, G or F , hence, we can consider six different possibilities for the first two morphisms ζ_1, ξ_1 :

- (1) $\zeta_1 \in M, \xi_1 \in M$;
- (2) $\zeta_1 \in F, \xi_1 \in F$;
- (3) $\zeta_1 \in G, \xi_1 \in G$;
- (4) $\zeta_1 \in M, \xi_1 \in F$ or vice versa;
- (5) $\zeta_1 \in M, \xi_1 \in G$ or vice versa;
- (6) $\zeta_1 \in G, \xi_1 \in F$ or vice versa.

We also can assume that no two consecutive morphisms ζ_i, ξ_j are from the same class and a is not from \mathbf{t} .

Case 1. Let $\psi : a \rightarrow \tau$ be a (unique) morphism in \mathbf{T}^{T+2} . Then one always can complete the span

$$a' \xleftarrow{\zeta_1} a \xrightarrow{\psi} \tau$$

to a commutative triangle by a morphism from M :

$$a' \rightarrow \tau.$$

Replacing ζ_1 by $\xi_1 : a \rightarrow a''$ and doing the same thing we see that the problem now is to prove that one can connect $\zeta_2 \cdot \dots \cdot \zeta_n : a' \rightarrow b$ and $\psi : a' \rightarrow \tau$ in the comacategory a'/\mathbf{t} and $\xi_2 \cdot \dots \cdot \xi_n : a'' \rightarrow c$ and $\phi : a'' \rightarrow \tau$ in a''/\mathbf{t} . But the ζ_2 and ξ_2 are from F or G so we are in the situation of cases 4 or 5.

Cases 4 and 5. Both cases are similar and are based on the following observation: any span

$$a' \xleftarrow{\phi} a \xrightarrow{\psi} a''$$

in \mathbf{T}^{T+2} , where $\psi \in M$ and $\phi \in F$ ($\phi \in G$) can be completed to a commutative square by a cospan

$$a' \xrightarrow{\psi^*} a''' \xleftarrow{\phi^*} a''$$

where $\psi^* \in M$ and $\phi^* \in F$ ($\phi^* \in G$.) This can be proved easily because the morphisms from F (from G) simply replace some number of K -edges by L -edges (X -edges). On the other hand the morphisms from M do not touch K or L -edges. So, we can apply ψ and then replace K -edges by L -edges (X -edges) or first replace the corresponding edges and then apply the same generating morphism (which we denote ψ^* .)

Applying this observation to the morphisms

$$a' \xleftarrow{\zeta_1} a \xrightarrow{\xi_1} a''$$

in the case 4 or 5 and taking into account that the morphisms from F (G) decrease the number of K -edges one conclude by induction that it is enough to prove our lemma in the cases 2,3 and 6.

Cases 2,3,6. As in case 1 let $\psi : a \rightarrow \tau$ be a (unique) morphism in \mathbf{T}^{T+2} . Then we can apply our observation from the cases 4 and 5 to the spans

$$a' \xleftarrow{\zeta_1} a \xrightarrow{\psi} \tau$$

and

$$\tau \xleftarrow{\psi} a \xrightarrow{\xi_1} a''$$

and then use the induction. □

8.4. Canonical filtration. For an object a from \mathbf{t} let say that a is of type (p, q) if it contains exactly p K -edges and q L -edges. For $k \geq 0$ let \mathbf{t}^k be the full subcategory of \mathbf{t} of objects of the type (p, q) $p + q \leq k$. Let also \mathbf{q}^k be the full subcategory of the objects of the type (p, q) $p + q = k, p \neq 0$ and let \mathbf{l}^k be the full subcategory of the objects of the type $(0, k)$ and let \mathbf{w}^k be the full subcategory of the objects of the type $(p, k), p + q = k$.

Lemma 8.5. • Each connected component of \mathbf{w}^k is isomorphic to the poset of subsets of the set $\{1, \dots, k\}$ (so it is a cube).

- Each connected component of \mathbf{q}^k is isomorphic to the poset of proper subsets of the set $\{1, \dots, k\}$ (so it is a punctured cube).

- The category \mathbf{l}^k is discrete and each connected component of \mathbf{l}^k is the maximal element in one of the connected components of \mathbf{w}^k .

Proof. The proof is an easy exercise. □

Proposition 8.6. *Let X be a functor $\mathbf{T}^{\mathbf{T}_\varepsilon, \mathbf{g}} \rightarrow \mathcal{E}$, where \mathcal{E} is a cocomplete category and let X_k be its restriction to \mathbf{t}^k . Then*

$$P = \operatorname{colim}_{\mathbf{T}^{\mathbf{T}_\varepsilon, \mathbf{g}}} X \simeq \operatorname{colim}_{\mathbf{t}} X_\infty = \operatorname{colim}_k P_k,$$

where

$$P_k = \operatorname{colim}_{\mathbf{t}^k} X_k.$$

Moreover, there is a pushout

$$(16) \quad \begin{array}{ccc} Q_k & \xrightarrow{w_k(X)} & L_k \\ \downarrow & & \downarrow \\ P_{k-1} & \longrightarrow & P_k \end{array}$$

where Q_k, L_k are colimits of the restriction of X on \mathbf{q}^k and \mathbf{l}^k respectively.

Proof. The first statement follows from Lemma 8.3.

There is an obvious functor $\mathbf{q}^k \rightarrow \mathbf{l}^k$ generated by the morphisms from F . It induces the morphism $Q_k \rightarrow L_k$.

Let us also construct a morphism $Q_k \rightarrow P_{k-1}$. We use the induction by k . The category \mathbf{t}^0 is trivial (as an object of \mathbf{Cat}/I) and consists of one object t_i for each $i \in I$. Hence, $P_0 = X$. The category \mathbf{q}^1 is discrete, hence Q_1 is a coproduct of the values of X on the objects $w \in \mathbf{q}^1$. These objects do not contain L -edges. Therefore, they belong to the same connected component of \mathbf{t}^1 . The terminal object of this connected component is t_i again and we define $Q_1 \rightarrow P_0$ on a summand $X(w)$ to be equal the value of X on the unique morphism $w \rightarrow t_i$.

Suppose now a morphism $Q_s \rightarrow P_{s-1}$ is constructed for any $m < k$ in such a way that P_s is the pushout from the statement of the Proposition.

Let a be an object of \mathbf{q}^k . Then it is also an object in $\mathbf{T}^{\mathbf{T}_\varepsilon}$ and so there is a terminal object $j(a)$ in the connected component of a in $\mathbf{T}^{\mathbf{T}_\varepsilon}$ and a unique morphism

$$\xi_a : a \rightarrow j(a)$$

in $\mathbf{T}^{\mathbf{T}_\varepsilon}$.

Notice also that $j(a) \in \mathbf{t}^{k-1}$ because a contains at least one K -edge and, therefore, there is, at least, one nonidentity morphism in $\mathbf{T}^{\mathbf{T}_\varepsilon}$ from a which belongs to G . But any morphism from G strictly decreases the type of the object. So, $j(a)$ belongs to one of \mathbf{l}^s , $s \leq k-1$. Now we define a morphism $X(a) \rightarrow P_{k-1}$ as the composite

$$X(a) \xrightarrow{X(\xi_a)} X(j(a)) \xrightarrow{i_a} L_s \rightarrow P_s \rightarrow P_{k-1},$$

where i_a is the canonical coprojection.

To prove that this construction defines a morphism $Q_k \rightarrow P_{k-1}$ we have to prove the commutativity of the following diagram

$$(17) \quad \begin{array}{ccccccc} X(a) & \longrightarrow & X(j(a)) & \longrightarrow & L_s & \longrightarrow & P_s \\ X(f_v) \downarrow & & & & & & \downarrow \\ X(b) & \longrightarrow & X(j(b)) & \longrightarrow & L_{s+1} & \longrightarrow & P_{s+1} \end{array}$$

where $a \in \mathbf{q}^k$ and has a type (p, s) , $p + s = k$, $p > 1$. The object b is of the type $(p - 1, s + 1)$ and f_v is a generator, which replaces a K -edge v of a by an L -edge. It implies that $s = k - p < k - 1$.

We will do it by exhibiting the diagram above as a composite of two commutative diagrams, one of which will commute by functoriality and another by our inductive assumption.

Observe that the morphism $\xi_a : a \rightarrow j(a)$ can be factorized as

$$a \xrightarrow{g} a' \xrightarrow{g_v} a'' \xrightarrow{m} j(a),$$

where $g \in G$ (replaces all K -edges by X -edges except the K -edge v), g_v is a generator which replaces K -edge v by an X -edge, and $m \in M$.

This show that the morphism $\xi_b : b \rightarrow j(b)$ can be factorized as

$$b \xrightarrow{g'} b' \xrightarrow{m'} j(b),$$

where $g' \in G$ (replaces all K -edges by X -edges) and $m' \in M$ and, moreover, the following diagram commutes

$$\begin{array}{ccc} a & \xrightarrow{g} & a' \\ f_v \downarrow & & \downarrow f'_v \\ b & \xrightarrow{g'} & b' \end{array}$$

The morphism $f'_v \in F$ is a generator which replaces the K -edge v by an L -edge.

Using the commutativity of the morphisms from F and M we construct the following commutative diagram

$$\begin{array}{ccc} a' & \xrightarrow{m''} & a''' \\ f'_v \downarrow & & \downarrow f''_v \\ b & \xrightarrow{m'} & j(b) \end{array}$$

Observe, that $a''' \in \mathbf{q}^{s+1}$ because it contains only one K -edge and replacing this edge by an L -edge (via f''_v) we end with an object from \mathbf{l}^{s+1} .

Using the observation from Lemma 8.3 we also have a commutative diagram

$$\begin{array}{ccc} a' & \xrightarrow{g_v} & a'' \\ m'' \downarrow & & \downarrow m''' \\ a''' & \xrightarrow{g'_v} & c \end{array}$$

and a commutative triangle

$$\begin{array}{ccc}
a'' & \xrightarrow{m} & j(a) \\
m''' \downarrow & \nearrow \mu & \\
c & &
\end{array}$$

where $\mu \in M$ because $j(a)$ is terminal in the connected component of $\mathbf{T}^{\mathbf{T}_g}$ and c does not contain K -edges.

The diagram

$$\begin{array}{ccccccc}
X(a''') & \longrightarrow & X(c) & \longrightarrow & X(j(a)) & \longrightarrow & P_s \\
\downarrow X(f_v'') & & & & & & \downarrow \\
X(j(b)) & \longrightarrow & L_{s+1} & \longrightarrow & P_{s+1} & &
\end{array}$$

commutes because $a''' \in \mathbf{q}^{s+1}$ and $s < k - 1$, so the induction assumption is applicable.

Putting all these morphisms together we have finally a commutative diagram

$$\begin{array}{ccccccccccc}
& & & & X(a'') & & & & & & \\
& & & & \nearrow & & \searrow & & & & \\
X(a) & \longrightarrow & X(a') & \longrightarrow & X(a''') & \longrightarrow & X(c) & \longrightarrow & X(j(a)) & \longrightarrow & P_s \\
\downarrow & & \downarrow & & \downarrow & & & & & & \downarrow \\
X(b) & \longrightarrow & X(b') & \longrightarrow & X(j(b)) & \longrightarrow & L_{s+1} & \longrightarrow & P_{s+1} & &
\end{array}$$

which proves the commutativity of (17).

The pushout property of P_k can be easily checked by comparing the universal properties of the colimit over \mathbf{t}^k and the pushout (16).

□

8.7. Calculating pushouts along free maps. Proposition 8.6 and Lemma 8.5 allow us to compute colimits over $\mathbf{T}^{\mathbf{T}_g}$ as sequential colimits of pushouts along the morphisms $w_k(X) : Q_k \rightarrow L_k$.

Let $(\mathcal{E}, \otimes, e)$ be a cocomplete symmetric monoidal category. And let $A = (X, K, L, f, g)$ be a data for pushout along a free map in the category of T -algebras. By Theorem 6.13 the underlying map $X \rightarrow P$ of the pushout can be calculated as the colimit of the functor $\tilde{A} : \mathbf{T}^{\mathbf{T}_g} \rightarrow \mathcal{E}$ which due to Proposition 8.6 can be replaced by the sequential colimit where each morphism is a pushout along a morphism $w_k(\tilde{A})$. The final step is to see that the morphism $w_k(\tilde{A})$ is a coproduct $\sqcup_i Y_i \otimes f_i$, where f_i are iterated canonical morphisms from the pushout-product axiom.

We use Lemma (8.5) and formula (11). Accordingly the functor \tilde{A} on an object $a \in \mathbf{t}$ is given by

$$(18) \quad \tilde{A}(a) = (\otimes_{v \in \chi(a)} X(v)) \otimes (\otimes_{v \in \kappa(a)} K(v)) \otimes (\otimes_{\lambda \in l(a)} L(v)).$$

Here, $\chi(a)$ is the set of X -edges in a , $\kappa(a)$ is the set of K -edges in a , and $\lambda(a)$ is the set of L -edges in a . The generating morphisms from q_S^k acts on \tilde{A} as tensor products $Y \otimes f_v$ where $f_v : K_v \rightarrow L_v$.

8.8. Model structure for algebras of polynomial monads. Let now \mathcal{E} be a monoidal model category and I be a set. Then the category \mathcal{E}/I has a model structure in which fibrations, cofibrations and weak equivalences are defined pointwise. Let K be the class of pointwise \otimes -cofibrations.

Theorem 8.9. *Let $(\mathcal{E}, \otimes, e)$ be a compactly generated monoidal model category and T be a tame polynomial monad. Then \mathcal{E}/I is K -compactly generated model category and the monad on \mathcal{E}/I generated by T is:*

- (a) *relatively K -adequate if \mathcal{E} satisfies the monoid axiom;*
- (b) *K -adequate if \mathcal{E} is strongly h -monoidal.*

Proof. The proof uses formula (18) and closely follows the proof of the Theorem 3.1. The only difference is that in general we have a coproduct of maps $\sqcup_i Y_i \otimes f_i$ we use in Theorem 3.1. This does not affect the argument for (a) because every saturated class of morphisms is closed under arbitrary coproducts. We use Proposition 1.15 for (b) and (c). □

Corollary 8.10. *Let $(\mathcal{E}, \otimes, e)$ be a compactly generated monoidal model category and T be a tame polynomial monad. Then the category $\text{Alg}_T(\mathcal{E})$:*

- (a) *admits a relatively left proper transferred model structure if \mathcal{E} satisfies the monoid axiom;*
- (b) *this model structure is left proper if \mathcal{E} is strongly h -monoidal .*

9. QUILLEN ADJUNCTION INDUCED BY A MAP OF POLYNOMIAL MONADS

The theory developed also allows to understand better functors between algebras generated by maps of polynomial monads. Let $\Phi : S \rightarrow T$ be a map of polynomial monads. Assume that \mathcal{E} is such that algebras of T and S admit transferred model structures, then we have a restriction functor δ^Φ and its left adjoint γ^Φ and this is a Quillen adjunction. Let also suppose that \mathcal{E} is a simplicial model category and its simplicial hom functor $S_{\mathcal{E}}(-, -)$ satisfies

$$S_{\mathcal{E}}(X, Y) \simeq S_{\mathcal{E}}(e, \mathcal{E}(X, Y)).$$

Then $\text{Alg}_T(\mathcal{E})$ and $\text{Alg}_S(\mathcal{E})$ are also simplicially enriched. The corresponding simplicial hom is given by a standard end (see [4], for example). We are interested in a formula for the total left derived functor of γ^Φ .

Theorem 9.1. *Suppose that \mathcal{E} has a symmetric lax-monoidal cofibrant replacement functor Q . Then the underlying I -collection of $\mathbb{L}_{\gamma^\Phi}(X)$ can be calculated as the following colimit:*

$$(19) \quad \mathbb{L}_{\gamma^\Phi}(X)(i) = \text{hocolim}_{\mathbf{b} \in \mathbf{T}^{\mathbf{s}(i)}} \widetilde{QX}(\mathbf{b})$$

where QX is the pointwise cofibrant replacement.

Proof. If Q is a symmetric lax-monoidal functor then pointwise application of Q to X gives a pointwise cofibrant S -algebra QX weakly equivalent to X . One can take then the bar-construction $B(S, S, QX)$ which will be a cofibrant replacement of X by general properties of bar-construction [4][Theorem 5.5] due to Corollary 6.14. To calculate the total left derived functor of γ^Φ we have to compute the underlying object of $\gamma^\Phi(B(S, S, QX))$. Since γ^Φ is a left adjoint this is equivalent to the simplicial realisation of a simplicial object

$$(20) \quad \gamma^\Phi(S(QX)) \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \gamma^\Phi(S^2(QX)) \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \gamma^\Phi(S^3(QX)) \begin{array}{c} \leftarrow \\ \rightleftarrows \\ \rightarrow \end{array} \cdots$$

By Theorem 5.14

$$\gamma^\Phi(S^k(QX)) = \operatorname{colim}_{\mathbf{T}^S} U_T(\widetilde{S^k(QX)})$$

Now we apply Theorem 5.15 to the following composite of morphisms of polynomial monads

$$Id_J \xrightarrow{\eta} S \xrightarrow{\Phi} T,$$

where η is the unit of the monad S . For an S -algebra Y we then have

$$S(Y) = F_S U_S(Y) = \gamma^\eta \delta^\eta(Y)$$

and by Theorem 5.15

$$U_T(\widetilde{S(Y)}) = U_T(\gamma^\eta \delta^\eta(Y)) = \operatorname{Lan}_{U_T(\zeta)} U_T(\widetilde{\delta^\eta(Y)}) = \operatorname{Lan}_{U_T(\zeta)} U_T(\zeta)^* U_T(\widetilde{Y}).$$

Hence, we have a formula

$$U_T(\widetilde{S^k(QX)}) = (\operatorname{Lan}_{U_T(\zeta)} U_T(\zeta)^*)^k U_T(\widetilde{QX}).$$

It is not hard to see that the categorical T -algebra $\mathbf{T}^{\operatorname{Id}_J}$ has the same objects as \mathbf{T}^S but only identity morphisms. Hence, for a functor $Z : \mathbf{T}^S \rightarrow V$

$$(\operatorname{Lan}_{U_T(\zeta)} U_T(\zeta)^*)^k(Z)(\mathbf{b}) = \sqcup_{\mathbf{b} \leftarrow \mathbf{b}' \leftarrow \dots \leftarrow \mathbf{b}_k} Z(\mathbf{b}_k),$$

So, the simplicial realization of the underlying object of the simplicial algebra (20) is given by the usual Bousfield-Kan simplicial replacement formula for homotopy colimit of \widetilde{QX} . □

Corollary 9.2. *The simplicial set $N(\mathbf{T}^S)$ is a cofibrant T -algebra in the category of simplicial set. In fact*

$$N(\mathbf{T}^S) = \mathbb{L}\gamma^\Phi(1),$$

where 1 is the terminal simplicial S -algebra.

Remark 9.3. Theorem 9.1 and Corollary 9.2 generalise formulas from [4] and [5] for the derived functor of symmetrisation of n -operads.

Part 3. Examples

One can show that categories of properads, dioperads, cyclic, modular operads, permutads, PROPs, wheeled PROPs and other categories of generalised operads based on graphs (see [13, 37, 36]) are all examples of categories of algebras of polynomial monads. In this part we look at some important examples in order to understand when corresponding polynomial monads are tame. We pay a particular attention to the n -operad case which turns out the most difficult case to prove tameity. For terminology on graphs and trees we refer the reader to the Appendix.

10. MONOIDS

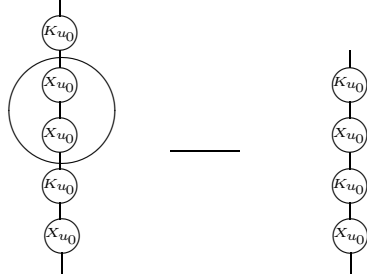
Let M be free monoid monad on Set . It is polynomial monad. The polynomial which generates this monad is:

$$1 \xleftarrow{s} N^* \xrightarrow{p} N \xrightarrow{t} 1.$$

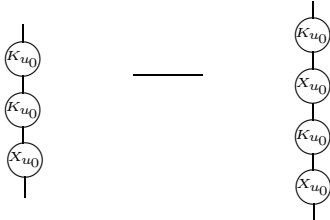
Here N is the set of natural numbers, which we can represent as a set of linear trees, that is trees where each vertex has valency two. The set N^* is the set of linear trees with one vertex marked. The map p forgets marking. Multiplication is the insertion of linear trees to the vertices of the target linear trees.

It is not difficult to describe explicitly the categories \mathbf{M}^{M+1} , and the final subcategory $\mathbf{m} \subset \mathbf{M}^{M,s}$ involved in the Schwede-Shipley construction.

The objects of \mathbf{M}^{M+1} , are decorated linear trees with vertices decorated by 0-ordinals and coloured by colours X or K . Morphisms are generated by contraction of a subtree whose all vertices have X -colours to a subtree with a single vertex coloured by X :



and replacing an edge with a tree with a single X -coloured vertex:



We write U_0 for the unique nonempty 0-ordinal. This extra decoration is, of course, not necessary but it will play a role for n -operads $n \geq 2$.

Obviously, in every connected components of \mathbf{M}^{M+1} there exists a terminal object which is a decorated linear tree whose vertices colours are alternating starting with X and terminating with X :



Hence, the free monoid monad is tame and we obtain, in particular, the formula (1) from the Introduction.

The objects from \mathbf{m} are, therefore, decorated linear trees with three colours of vertices X, K, L such that

- the first and the last vertices (in the natural order generated by linearity of the underlying tree) are coloured by X ;
- no two vertices of the same colours are connected by an edge;
- no two vertices with the colours K and L are connected by an edge.

Let $A = (R, Y_0, Y_1, u, \alpha)$ be a data for pushout along a free map in the category of monoids (see the proof of Theorem 3.1). Then, according to the formula (18) the functor \tilde{A} on a typical object from \mathbf{m} is isomorphic to the product

$$R \otimes Y_{i_1} \otimes R \otimes Y_{i_2} \otimes R \otimes \dots \otimes Y_{i_n} \otimes R,$$

where (i_1, \dots, i_n) is the vertex of the punctured cube, and we arrive to the formula (3) (see also the formula for $W(S)$ on p.10 in [46]).

11. NON-SYMMETRIC OPERADS

A 1-operad is the same as classical nonsymmetric operad.

The free nonsymmetric operad monad is polynomial and is represented by

$$\mathbf{RCor}(1) \leftarrow \mathbf{RTr}^*(1) \rightarrow \mathbf{RTr} \rightarrow \mathbf{RCor}(1)$$

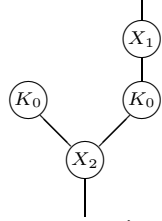
Here $\mathbf{RCor}(1)$ is the set of isomorphism classes of corollas (which is isomorphic to the set of natural numbers). The set $\mathbf{RTr}(1)$ is the set of isomorphism classes of planar rooted trees. The elements of the set $\mathbf{RTr}^*(1)$ are the elements from $\mathbf{RTr}(1)$ with one vertex marked. And the middle map forgets the marking. The target map associates to $S \in \mathbf{RTr}(1)$ the result of its total contraction $t(S)$ and the source map associates to an $S \in \mathbf{RTr}^*(1)$ with the marked vertex v the corolla $cor_v(S)$. Multiplication is the insertion of a planar tree to the vertices of the target tree.

The proof that the free nonsymmetric operads is tame is now very similar to the case of monoids.

Let $O(1)$ be the free nonsymmetric operad monad. Then the objects of $\mathbf{O}(1)^{0(1)+1}$ are (isomorphism classes) of planar trees whose vertices are coloured by X and K . As in the monoid case a decoration by 1-ordinals is a redundant information because it can be reconstructed from the planar structure of the tree.

Morphisms in $\mathbf{O}(1)^{0(1)+1}$ are generated by contractions of a subtree whose all vertices have X -colours to a subtree with a single vertex coloured by X and by replacement of an edge with a tree with a single X -coloured vertex. A typical terminal object in a connected component of $\mathbf{O}(1)^{0(1)+1}$ is a planar coloured tree such that no two vertices of the same colours are connected by an edge. We also

require that the colour of a vertex which has a root as its outcoming edge or a leave as its incoming edge is X . For example, a tree of this form is a terminal object in its connected component:



Here we use natural numbers as notations for 1-ordinals.

For more details for model category structure on nonsymmetric operads see [41].

12. SYMMETRIC OPERADS

The free symmetric operad monad is represented by the polynomial

$$\mathbf{RCor}(1) \leftarrow \mathbf{RTr}^* \rightarrow \mathbf{RTr} \rightarrow \mathbf{RCor}(1).$$

The set \mathbf{RTr} is the set of isomorphism classes of ordered rooted trees. It is not hard to see that such an isomorphism class can be represented by a planar tree together with a permutation of its leaves, or in other words, planar tree with chosen linear order. The structure maps of the above polynomial are defined similarly to the nonsymmetric operad case.

One can define a polynomial monad whose algebras are *reduced symmetric operads*. They are symmetric operads without 0-space. The corresponding polynomial is

$$\mathbf{RCor}_{rd} \leftarrow \mathbf{RTr}_{rd}^* \rightarrow \mathbf{RTr}_{rd} \rightarrow \mathbf{RCor}_{rd}.$$

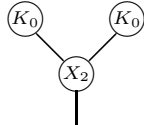
Everything is defined similarly to the above except that our trees and corollas should not have vertices without incoming edges (i.e. stamps).

Finally, one can define a polynomial monad whose algebras are *normal symmetric operads*. This are reduced symmetric operads whose unary operations is 1. A reduced tree is called *normal* if every vertex has at least two incoming edges. We have a polynomial representing free normal symmetric operad

$$\mathbf{RCor}_{nrm} \leftarrow \mathbf{RTr}_{nrm}^* \rightarrow \mathbf{RTr}_{nrm} \rightarrow \mathbf{RCor}_{nrm}.$$

Free reduced symmetric operad monad is tame. Implicitly, this was first observed by Getzler and Jones in [23][Section 1.5]. As in nonsymmetric operad case one can easily characterise the terminal objects in connected components of corresponding internal algebra classifier as alternating trees with two colours X and K (see Section 11). Free normal symmetric operad monad is also tame.

Free symmetric operad monad is not tame. The following tree



is the only candidate for a terminal object in one of the connected components of corresponding internal algebra classifier but it has a nontrivial automorphism coming from an action of \mathbb{S}_2 on X_2 , so, it is not terminal. One can construct an obstruction for the existence of model structure on symmetric operads with

coefficient in chain complexes over a field of positive characteristic similarly to (15.24). In fact, this argument was first proposed by Benoit Fresse in the symmetric operad settings [19].

13. CYCLIC AND MODULAR OPERADS.

In this section we show how to construct a polynomial monads for modular cyclic and planar cyclic operads, modular operads and consider question of tameity of these monads.

The constructions of all these monads are similar to the case of symmetric operads. We simply have to use different categories of graphs described in the Appendix of our paper. In order not to repeat the same construction again and again we do it for modular operads case leaving the other cases for the reader.

The free modular operad monad $O(Md)$ is represented by the polynomial

$$\mathbf{Cor} \leftarrow \mathbf{Gr}_c^* \rightarrow \mathbf{Gr}_c \rightarrow \mathbf{Cor}.$$

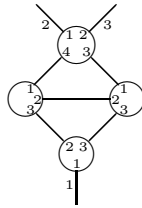
Here \mathbf{Cor} is the set of isomorphism classes of corollas. The set \mathbf{Gr}_c is the set of isomorphism classes of ordered connected graphs with nonempty set of vertices and \mathbf{Gr}_c^* is the set of isomorphism classes of ordered connected graphs with nonempty set of vertices and with one vertex marked. The target and source maps and composition operations are defined similarly to the previous cases.

For cyclic operad monad $O(Cl)$ we have to replace \mathbf{Gr}_c by the set of isomorphism classes of ordered trees for planar version of cyclic operad monad $O(Cl)$ we replace it by the set of isomorphism classes of planar trees. Neither of these monads are tame for the reason similar to the case of free symmetric operad monad. Nevertheless, some reduced version of $O(Cl)$ and $O(Cl)$ are tame. We call a cyclic (planar cyclic) operad *reduced* if its set of operations which arity is corolla with exactly one edge is empty.

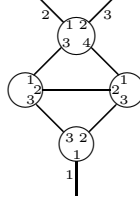
Proposition 13.1. *The monad for free reduced cyclic (planar cyclic) operad $O(Cl)_{rd}$ ($O(Pcl)_{rd}$) is tame polynomial.*

Proof. It is clear how to construct polynomials for these monads. Again, it is not hard to see that morphisms in the category $\mathbf{O}(\mathbf{Cl})_{rd}^{0(Cl)_{rd}+1}$ are generated by contractions of trees with two colours and substitution of corollas to the vertices which can be simply understood as reordering of the set of incoming edges for vertices of the tree. The terminal objects in the connected components can be characterised as alternating coloured trees. Similarly for planar case. \square

Unfortunately, the polynomial monad for modular operads is not tame even if we restrict themselves to the reduced operads without units. Indeed, in the corresponding internal algebra classifier will morphism are generated by contraction of subgraphs and substitution of corollas to the vertices. In particular, there is a connected component which contains the following ordered bicoloured graph:



in which top and bottom vertices have colour X but middle vertices have colour K . The isomorphism class of this graph can not be contracted further but it admits a nontrivial automorphism generated by renumbering of the incoming edges in the X -vertices as shown in the following picture:



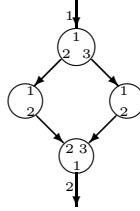
So, we can not hope to have a model structure on modular operads under general assumption of our main Theorem.

14. PROPERADS, PROPS AND WHEELED PROPS

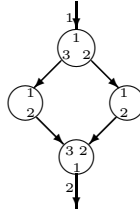
The definitions of polynomial monad for properads, PROPs and wheeled versions of them are very similar to the previous. For PROPs (properads) we simply use oriented loop free (connected) graphs. For wheeled versions we use all oriented (connected) graphs. We also have reduced properads and PROPs and reduced wheeled properads and wheeled PROPs, where we require that the sets of operations of the type $A(0, n)$ and $A(m, 0)$ are empty.

Proposition 14.1. *The monads for free properads, wheeled properads, PROPs, wheeled PROPs as well as the monads for their reduced versions are not tame.*

Proof. Nonreduced case is obvious because all these categories contain the category of symmetric operad. To show that monad for reduced properads, reduced wheeled properads and PROPs are not tame we observe that the following graph lives in one of the connected components of \mathbf{T}^{T+1} where T is one of these monads:

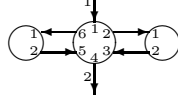


in which top and bottom vertices have colour X but middle vertices have colour K . As in modular operad case the isomorphism class of this graph can not be contracted further but it admits a nontrivial automorphism generated by renumbering of the incoming edges in the X -vertices as shown:



The case of reduced wheeled PROPs is different since in the corresponding internal algebra classifier \mathbf{T}^{T+1} graphs admit further contractions since we are no longer restricted by the requirement of nonexistence of direct loops in the graphs

(see [38][Remark after Def.2.1.8]). So we can multiply X -vertices (something which is not allowed in properads) and contract any edges between X -vertices (which is not allowed in PROPs). Doing this operation we end up with a graph which has only one X -vertex connected to K -vertices. Nevertheless the following graph, which does not admit further contraction, has a nontrivial automorphism in \mathbf{T}^{T+1} :



In this graph the central vertex has colour X all the others have colour K . The nontrivial automorphism is generated by substitution which interchanges 2 and 6 and 5 and 3 inside the X -vertex.

□

15. n -OPERADS

15.1. Complementary relations and n -ordinals.

Definition 15.2. Let $0 \leq n \leq \infty$. Let X be a set equipped with a family of n binary relations

$$<_0, \dots, <_{n-1}.$$

These relations are called *complementary* if they satisfy the following properties:

- (1) $<_p$ is nonreflexive for any $0 \leq p \leq n-1$;
- (2) for every pair $a, b \in X$, there exists exactly one p such that

$$a <_p b \text{ or } b <_p a;$$

Definition 15.3. A finite set T equipped with n complementary relations

$$<_0, \dots, <_{n-1}.$$

is called an n -ordinal if the following transitivity property holds:

- if $a <_p b$ and $b <_q c$ then $a <_{\min(p,q)} c$.

Let X and Y be two sets equipped with n complementary relations.

Definition 15.4. A relation preserving morphism

$$\sigma : X \rightarrow Y$$

is a map $\sigma : X \rightarrow Y$ of underlying sets such that

$$i <_p j \text{ in } X$$

implies that

- (1) $\sigma(i) <_r \sigma(j)$ for some $r \geq p$ or
- (2) $\sigma(i) = \sigma(j)$ or
- (3) $\sigma(j) <_r \sigma(i)$ for $r > p$.

in Y .

Sets with n complementary relations form a category Rel_n^c with respect to the relation preserving morphisms. There is a functor $|-| : Rel_n^c \rightarrow Set$ which forgets relations.

There are two interesting subcategories of Rel_n^c . First, consider a subcategory D of Rel_n^c of morphisms σ such that $|\sigma| = id$. In this subcategory there is at most one morphism between two objects. So, D is a (large) poset.

Definition 15.5. Let X, Y be two sets with n complementary relations. We will say that X dominates Y if there is a morphism $X \rightarrow Y$ in D .

Another subcategory of Rel_n^c is the full subcategory $Ord(n)$ of n -ordinals. As there are no nontrivial automorphisms in $Ord(n)$ we will assume that each isomorphism class of n -ordinals contains a single element (i.e. $Ord(n)$ is skeletal).

Every n -ordinal can be represented as a pruned planar tree with n levels (pruned n -tree) or as n -dimensional globular graph (see [4] for a discussion). The empty n -ordinal is represented by the only degenerate pruned n -tree $z^n U_0$ which consists of only a roote on the level 0. The terminal n -ordinal is represented by a linear tree U_n (or just an n -globe in globular notations).

For a k -ordinal R , $k \leq n$ we consider its $(n - k)$ -th vertical suspension $S^{n-k} R$ which is an n -ordinal with the underlying set R , and the order $<_m$ equal the order $<_{m-k}$ on R (so $<_m$ are empty for $0 \leq m < n - k$.)

The vertical suspension provides us with a functor $S : Ord(n) \rightarrow Ord(n + 1)$. We also define an ∞ -suspension functor $Ord(n) \rightarrow Ord(\infty)$ as follows. For an n -ordinal T its ∞ -suspension is an ∞ -ordinal $S^\infty T$ whose underlying set is the same as the underlying set of T and $a <_p b$ in $S^\infty T$ if $a <_{n+p-1} b$ in T . It is not hard to see that the sequence

$$Ord(0) \xrightarrow{S} Ord(1) \xrightarrow{S} Ord(2) \longrightarrow \dots \xrightarrow{S} Ord(n) \longrightarrow \dots \xrightarrow{S^\infty} Ord(\infty),$$

exhibits $Ord(\infty)$ as a colimit of $Ord(n)$.

Maps of n -ordinals admit fibers in the following sense. Let $\sigma : T \rightarrow S$ be a map of n -ordinals and let $i \in S$. Then the set $\sigma^{-1}(i)$ inherits a structure of an n -ordinal from T .

Definition 15.6. A map of n -ordinals is called quasibijection if it is a bijection of the underlying sets.

Observe, that for each n -ordinal T the set $|T|$ has a natural linear order $<$. By definition

$$a < b \text{ if there exists } p \in \{0, \dots, n - 1\}, a <_p b.$$

We call this relations the total order on $|T|$. It follows from this that any map of n -ordinals $\sigma : T \rightarrow S$ comes with a linearly ordered set of its fibers.

Definition 15.7. A map σ of n -ordinals $1 \leq n \leq \infty$ is called order preserving if it preserves the total orders of the n -ordinals or, equivalently, only conditions 1 and 2 from the Definition 15.4 hold for σ .

Quasibijections and order preserving maps form a factorisation system in $Ord(n)$.

Lemma 15.8. [6] For every morphism $\sigma : T \rightarrow S$ in $Ord(n)$ $1 \leq n \leq \infty$ there exists a factorisation

$$T \xrightarrow{\pi} T' \xrightarrow{\nu} S$$

where π is a quasibijection, ν is order preserving and π is order preserving on fibers of ν .

Let \mathcal{E} be a symmetric monoidal category.

Definition 15.9. An n -collection in \mathcal{E} is a family of objects $A_T \in \mathcal{E}$ indexed by n -ordinals.

Let $\text{Coll}_n(\mathcal{E})$ be a category of n -collections and their levelwise morphisms.

We now recall the definition of pruned $(n-1)$ -terminal n -operad [4]. Since we do not need other types of n -operads in this paper we will call them simply n -operads. The notation U_n means the terminal n -ordinal.

For a morphism of n -ordinals $\sigma : T \rightarrow S$ the ordered list of n -ordinals $\{T_0, \dots, T_k\}$ is the list of fibers of σ . Analogously, for a composite

$$T \xrightarrow{\sigma} S \xrightarrow{\omega} R,$$

we have a list of fibers of composite and the list of fibers of fibers of ω . We believe that the notations in the definition below are self obvious.

Definition 15.10. An n -operad in \mathcal{E} is a collection A_T , $T \in \text{Ord}(n)$ of objects of \mathcal{E} equipped with the following structure :

- a morphism $\epsilon : e \rightarrow A_{U_n}$ (the unit);
- for every morphism $\sigma : T \rightarrow S$ in $\text{Ord}(n)$, a morphism

$$m_\sigma : A_S \otimes A_{T_0} \otimes \dots \otimes A_{T_k} \rightarrow A_T \quad (\text{the multiplication}).$$

They must satisfy the following identities:

- for any composite

$$T \xrightarrow{\sigma} S \xrightarrow{\omega} R,$$

the associativity diagram

$$\begin{array}{ccc}
 A_R \otimes A_{S_\bullet} \otimes A_{T_0^\bullet} \otimes \dots \otimes A_{T_i^\bullet} \otimes \dots \otimes A_{T_k^\bullet} & \simeq & A_R \otimes A_{S_0} \otimes A_{T_1^\bullet} \otimes \dots \otimes A_{S_i} \otimes A_{T_i^\bullet} \otimes \dots \otimes A_{S_k} \otimes A_{T_k^\bullet} \\
 \downarrow & & \downarrow \\
 A_S \otimes A_{T_1^\bullet} \otimes \dots \otimes A_{T_i^\bullet} \otimes \dots \otimes A_{T_k^\bullet} & & A_R \otimes A_{T_\bullet} \\
 \searrow & & \swarrow \\
 & A_T &
 \end{array}$$

commutes, where

$$A_{S_\bullet} = A_{S_0} \otimes \dots \otimes A_{S_k},$$

$$A_{T_i^\bullet} = A_{T_i^0} \otimes \dots \otimes A_{T_i^{m_i}}$$

and

$$A_{T_\bullet} = A_{T_0} \otimes \dots \otimes A_{T_k};$$

- for an identity $\sigma = \text{id} : T \rightarrow T$ the diagram

$$\begin{array}{ccc}
 A_T \otimes A_{U_n} \otimes \dots \otimes A_{U_n} & \xleftarrow{\quad} & A_T \otimes e \otimes \dots \otimes e \\
 \downarrow & \nearrow \text{id} & \\
 A_T & &
 \end{array}$$

commutes;

- for the unique morphism $T \rightarrow U_n$ the diagram

$$\begin{array}{ccc}
A_{U_n} \otimes A_T & \xleftarrow{\quad} & e \otimes A_T \\
\downarrow & \swarrow id & \\
A_T & &
\end{array}$$

commutes.

The notion of n -operad morphism is obvious and we have a category $O_n(\mathcal{E})$ of n -operads. We also have forgetful functor

$$U : O_n(\mathcal{E}) \rightarrow \text{Coll}_n(\mathcal{E}).$$

If \mathcal{E} is cocomplete then there exists a left adjoint to U , moreover U is monadic.

15.11. Reduced and normal n -operads.

Definition 15.12. A reduced n -collection in \mathcal{E} is a family of objects $A_T \in \mathcal{E}$ indexed by the set of nonempty n -ordinals $R\text{Ord}(n)$.

There is a category structure on $R\text{Ord}(n)$ which we will call the category of *reduced n -ordinals*. The morphisms are maps of n -ordinals which are surjective on the underlying sets. The fibers of such a map are necessary reduced. A *reduced n -operad* is defined similarly to n -operad but we use reduced n -collections and maps from $R\text{Ord}(n)$. We have the category of reduced n -collections $\text{RColl}_n(\mathcal{E})$ and reduced n -operads $RO_n(\mathcal{E})$.

We have a forgetful functor

$$RU : RO_n(\mathcal{E}) \rightarrow \text{RColl}_n(\mathcal{E}).$$

If \mathcal{E} is cocomplete then there exists a left adjoint to RU , and RU is monadic.

Definition 15.13. An n -ordinal is called *normal* if it is nonempty and not equal to the terminal n -ordinal.

Again, we have a category of *normalized n -ordinals* $N\text{Ord}(n)$ with morphisms being surjective morphisms between n -ordinals. But in the list of fibers of a map we include only fibers which are not equal to the terminal n -ordinal. We have the corresponding category of *normal n -collections* NColl_n and *normal n -operads* $NO_n(\mathcal{E})$. Obviously, $NO_n(\mathcal{E})$ is a full subcategory of $RO_n(\mathcal{E})$ consisting of reduced n -operads A such that $A_{U_n} = e$. We have a forgetful functor

$$NU : NO_n(\mathcal{E}) \rightarrow \text{NColl}_n(\mathcal{E}),$$

which is monadic provided \mathcal{E} is cocomplete.

15.14. Free normal n -operad monad. An explicit description of the free normal n -operad on $X \in \text{NColl}_n(\text{Set})$ is following [4]

$$(21) \quad NO_n(X)_S = \coprod_{\tau \in \text{Trees}_S(n)} \prod_{v \in \text{vertices}(\tau)} X_{T_v}$$

Here $S \in N\text{Ord}(n)$ is an n -ordinal and $\text{Trees}_S(n)$ is the set of *labelled decorated planar trees* which we now describe.

Let τ be a rooted tree with at least two incoming edges at each vertices equipped with:

- a structure of n -ordinal $T_v \in NOrd(n)$ (decoration) on the set of incoming edges of every vertex v ;
- a labeling of the set of its leaves that is a bijection between the set of leaves of τ and the set $|S|$.

Recall first that any decorated labeled tree of the form above determines an n -complementary relation on the set $|S|$ [4]. Let w be a vertex or a leave of τ and v be a vertex of τ . We will say that w is *above* v if there exists a path in τ from w to v which does not contain two consecutive input edges of the same vertex. For any two leaves or vertices there exists a unique vertex $v(k, l)$ which is below k, l and such that any other vertex below to k and l is below v .

The n -complementary relation on $|S|$ generated by τ is constructed as follows. For $p, q \in |S|$ let k, l be corresponding leaves on τ . Let e_p be the input edge in $v(k, l)$ which is the last edge in the path from k to $v(k, l)$. Analogously let e_q be the input edge in $v(k, l)$ which is the last edge in the path from l to $v(k, l)$. Let T be the n -ordinal which decorates $v(k, l)$. By definition $p <_r q$ if $e_p <_r e_l$.

A decorated planar tree τ belongs to $Trees_S(n)$ if it satisfies the following condition :

- * the complementary relation generated by τ on the set $|S|$ is dominated by S .

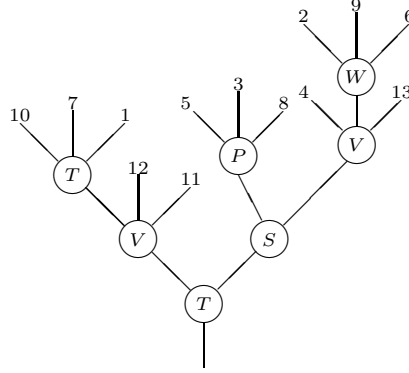


FIGURE 1. Typical labeled decorated tree. Here V, W, P, T, S are n -ordinals decorating corresponding vertices.

Proposition 15.15. *The free normal n -operad monad $NO(n)$ is polynomial.*

Proof. The following polynomial generates the monad $NO(n)$:

$$\mathbf{RCor}_{nrm}(n) \leftarrow \mathbf{RTr}_{nrm}^*(n) \rightarrow \mathbf{RTr}_{nrm}(n) \rightarrow \mathbf{RCor}_{nrm}(n)$$

Here $\mathbf{RCor}_{nrm}(n) = \cup_S Trees_S(n)$ is the set of isomorphism classes of planar corollas decorated by normalised n -ordinals (which is isomorphic to the set of normalised n -ordinals, of course). The set $\mathbf{RTr}_{nrm}(n)$ is the set of isomorphism classes of labelled decorated planar trees. The elements of the set $\mathbf{RTr}_{nrm}^*(1)$ are the elements from $\mathbf{RTr}_{nrm}(1)$ with one vertex marked. And the middle map forgets the marking. The target map associates to $\tau \in Trees_S(n) \subset \mathbf{RTr}_{nrm}(n)$ the corolla decorated by S and the source map associates to an $S \in \mathbf{RTr}^*(1)$ with the marked vertex v the corolla $cor_v(S)$ with its decoration. Multiplication in this monad is

given by insertion of labelled decorated planar trees to the vertices of the ambient tree and relabeling.

□

15.16. The categorical n -operad $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$. According to the formulas (8) and (21) the n -collection $Trees_S(n), S \in \mathbf{NO}(\mathbf{n})$ is the object of objects of the categorical n -operad $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$. The morphisms of $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$ are generated by surjections of normal n -ordinals in the following sense. Let $\sigma : T \rightarrow S$ be a surjection of normal n -ordinals. Let $\tau \in \mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$ be a decorated planar tree which contains a subtree τ' whose root is decorated by S and whose vertices adjacent to this root are decorated by fibers of σ . We contract this subtree to a corolla whose only vertex is decorated by T . We also relabel the leaves of τ according to the permutation induced by σ . This produces a decorated planar tree τ' . We will have a morphism $\tau \rightarrow \tau'$ in $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$ if τ' tree satisfies the condition (*). Relations between morphisms are coming from associativity relations in n -operads with respect to compositions of surjections.

We also can provide a smaller set of generators. By Lemma 15.8 the morphisms are generated by two classes of morphisms :

- Those which correspond to quasibijections on a given node. They do not change the underlying planar tree but change its labeling.

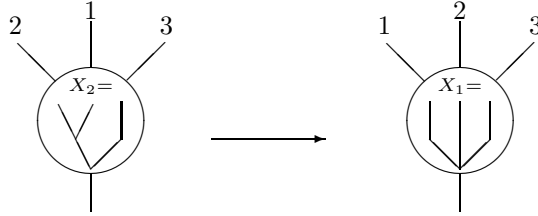


FIGURE 2. A generator which corresponds to a quasibijection: $\sigma : X_1 \rightarrow X_2, \sigma(1) = 2, \sigma(2) = 1, \sigma(3) = 3$.

- Contraction of an edge along an order preserving proper (i.e. not a quasibijection) surjections of n -ordinals. They do not change the labeling of the tree.

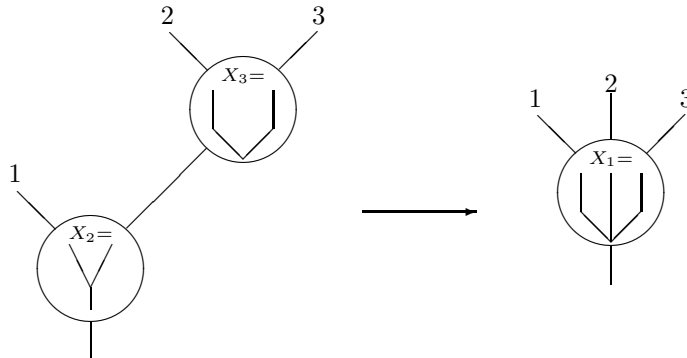


FIGURE 3. A generator which corresponds to an order preserving map:
 $\sigma : X_1 \rightarrow X_2$, $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 2$.

Lemma 15.17. *The category $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$ is a directed category in the sense that there exists a function l from objects of $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$ to the set of natural numbers such that the value of l on the target of any morphism is always greater (strictly for nonidentity morphisms) than the value of l on the source of the morphism.*

Proof. For an n -ordinal T define the *volume* $V(T)$ to be the sum of all edges of the n -tree representation of T . Let $\mathbf{in}(\tau)$ be the number of internal edges in the planar tree τ . For a decorated planar tree τ let

$$l(\tau) = \left(\sum_{v \in \text{vertices}(\tau)} V(T_v) \right) - n \times \mathbf{in}(\tau) - 1.$$

It is sufficient to check that l strictly increases from source to target of a generating morphism of $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$.

In a case of a generator coming from a quasibijection the statement is obvious because a quasibijection strictly decreases the volume of n -ordinals.

If a generator $f : \tau \rightarrow \tau'$ corresponds to a contraction of an edge along an order preserving map $T \rightarrow S$ with a unique nontrivial fiber T_i then it is not hard to prove that $V(S) + V(T_i) \geq V(T) + n$. Therefore, the function $l(\tau') - l(\tau) \geq 1$. See Figure 3 for an example.

□

15.18. Semifree coproducts of normal n -operads. We have to describe the category $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$ explicitly. According to (7.3) objects of this category consist of labeled decorated planar trees with an additional decoration of each vertices by colours X or K . We call such trees *coloured decorated trees*. The vertices of such a tree will be called X -vertices or K -vertices according to their colours.

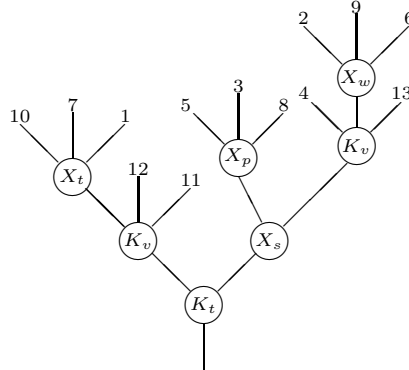


FIGURE 4. Typical coloured decorated tree. Here v, w, p, t, s are n -ordinals decorating corresponding vertices.

The generators of morphisms in $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$ can be described as follows. A generator corresponds to a morphism of n -ordinals and it contracts a part of the tree. In this contraction the K -vertices remain intact. Since these generators are the same as in $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$ we also can use as generators the morphisms which correspond to quasibijections and order preserving maps of n -ordinals with only one fiber similar to the case $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})}$.

Lemma 15.19. *In the category $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$:*

- (1) *Any two parallel generators are equal ;*
- (2) *Any span of generators*

$$a' \xleftarrow{\phi} a \xrightarrow{\psi} a''$$

can be completed to a commutative square by a cospan of generators (or identities)

$$a' \xrightarrow{\psi^*} a''' \xleftarrow{\phi^*} a''.$$

- (3) *The category $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$ is directed.*

Proof. Two parallel generators can be either

- (1) two quasibijections;
- (2) two order preserving proper surjections.

Since morphisms are parallel the first case is possible only if both quasibijections begin in the same node. But in this case these quasibijections are completely determined by the labeling of the leaves of the tree and decorations of vertices by n -ordinals. For the the second case observe that a proper order preserving surjection is completely determined by its list of fibers. This proves the first statement.

Let $a \in \mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$. With each generator out of a one can associate a one node subtree with an X -vertex in the underlying tree of a (if generator corresponds to a quasibijection) or a subtree with two X -vertices (if generator corresponds to an order preserving surjection). Let us call this subtree *the fiber* of the generator.

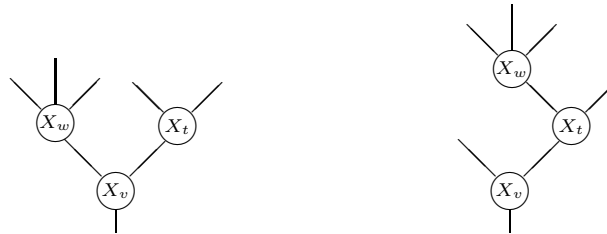
For the morphisms $a \rightarrow a'$ and $a \rightarrow a''$ there are, therefore, two possibilities:

- (1) The union (meaning the union of vertices and their input and output edges) of the fibers of these morphisms is a subtree in a ;
- (2) This union of the fibers is a disjoint union of subtrees.

In the second case the second statement of the Lemma is obvious.

The first case can be subdivided on several cases depending on the shape of the union of the fibers:

- (1) There are three vertices in this union:



- (2) There are two vertices in the union;

FIGURE 5. Two typical possibilities for a tree with three vertices.

- (3) There is exactly one vertex in the union.



FIGURE 6. A typical tree with two and one vertices.

The first case is clear because it means that both generators are contractions of the edges along proper order preserving surjections. The statement then amounts to an associativity condition for multiplication in n -operads.

Let us consider the third case. The two morphisms out of a correspond to a cospan of quasibijections:

$$Y \rightarrow W \leftarrow Z.$$

Let a belong to the category $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$ where S is an n -ordinal. We will construct an object a'' out of a by replacing the n -ordinal W by another n -ordinal T . Let i and j be two input edges in X_w . We define $i <_r j$ in T if $p <_r q$ for $p \in a^{-1}(i), q \in a^{-1}(j)$ in S . Here, $a^{-1}(i)$ is the set of leaves l of a such that i is an output edge of a vertex below l or i is a leaf and then $a^{-1}(i) = \{i\}$. We have to prove correctness of this definition because there could be many leaves above a given vertex. The correctness follows from the observation that the complementary order relation on the set of labels generated by a depends on the n -ordinal structure on the set of input edges of a vertex where the direct paths meet. Since S dominates a it follows that for any two elements from $a^{-1}(i)$ and $a^{-1}(j)$ their relation in S is the same. This complementary relation is also an n -ordinal.

Now, it is easy to check by comparing the corresponding complementary relations that there exists a span of quasibijections $Y \leftarrow T \rightarrow Z$ making the diagram commutative. So we can construct our cospan of morphisms in $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$:

$$a' \rightarrow a''' \leftarrow a''.$$

The remaining case is of similar nature.

Finally for the third statement of the Lemma we can use the function l from Lemma 15.17.

□

Proposition 15.20. *For every n -ordinal S each connected component of the category $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$ has a terminal object.*

Proof. First, we will prove that $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$ has a unique weak terminal object that is an object such that there exists at least one morphism to it from any other object of a connected component of $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$.

Given a zig-zag of morphisms in $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$

$$a_0 \leftarrow a_1 \rightarrow a_2 \leftarrow \dots \rightarrow a_{n-2} \leftarrow a_{n-1} \rightarrow a_n$$

by Lemma 15.19 one can replace it by a zig-zag

$$a_0 \rightarrow b_1 \leftarrow a_2 \leftarrow \dots \rightarrow a_{n-2} \rightarrow b_{n-1} \leftarrow a_n.$$

Doing the same for the zig-zag of b 's and continuing we come to the conclusion that for any two objects c, c' from the same connected component of $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$ one can find an object c'' and a cospan

$$c \rightarrow c'' \leftarrow c'.$$

We claim that a connected component of a finite directed category C with the above property has a weak terminal objects and this object is unique. We use an induction by the number of objects k . It is obviously true if $k = 1$. Now assume that it is true for $k = m-1$. One can assume that there is only one connected component, otherwise the statement is true by inductive assumptions. Let L be the minimum of the function l on C . Consider the full subcategory C' which consists of objects a such that $l(a) > L$. Then C' is obviously connected and satisfies our inductive requirements. Therefore, it contains a weak terminal object t . If an object a is not from C' then there must be a span

$$a \rightarrow b \leftarrow t$$

where $b \in C'$. In this span $b \leftarrow t$ must be an identity, otherwise $l(b) > l(t)$ and t can not be weakly terminal. So, we found a map from any object of C to t . This weak terminal object is obviously unique.

Next step of the proof is to show that the weakly terminal object t in a connected component of $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$ is actually terminal. We use simple induction by the values of l to prove that there is at most one morphism to t . Indeed, the statement is true for all objects a such that $l(a) \geq l(t)$. Now, suppose we know that the morphism is unique for all objects a such that $l(a) \geq m$. Let k be the maximal integer such that $k < m$ and there exists an object b such that $l(b) = k$. Let $l(b) = k$. and $f, g : b \rightarrow t$. One can factorise $f = f_1 \cdot f_2$ and $g = g_1 \cdot g_2$ where f_1 and g_1 are generators. Now, we can complete the cospan

$$a_1 \xleftarrow{f_1} b \xrightarrow{g_1} a_2$$

to a span

$$a_1 \rightarrow c \leftarrow a_2.$$

If $a_1 \neq a_2$ then we can use our inductive assumption and , therefore, there is only one morphism from c to t and we finished the proof. If $a_1 = a_2$ then $f_1 = g_1$ by Lemma 15.19 and $f_2 = g_2$ by inductive assumptions.

□

15.21. Free reduced n -operad monad. An explicit description of the free reduced n -operad on $X \in \mathbf{NColl}_n(\mathbf{Set})$ is similar to the normal case [4]:

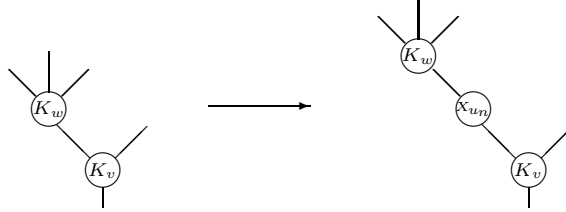
$$(22) \quad R(X)_S = \coprod_{\tau \in \mathbf{Trees}_S^r(n)} \prod_{v \in \text{vertices}(\tau)} X_{T_v}$$

Here $S \in \mathbf{ROrd}(n)$ is an n -ordinal and $\mathbf{Trees}_S^r(n)$ is the set of *reduced labelled decorated planar trees*. A reduced labelled decorated planar tree is defined similar

to the labelled decorated planar tree but we allow vertices with one incoming edge. The resulting monad $RO(n)$ is clearly polynomial. We have to prove now that it is tame.

Proposition 15.22. *For every n -ordinal $S \in ROrd(n)$ each connected component of the category $\mathbf{RO}(\mathbf{n})^{\mathbf{RO}(\mathbf{n})+1}$ has a terminal object.*

Proof. The category $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$ can be considered as a subcategory of $\mathbf{RO}(\mathbf{n})^{\mathbf{RO}(\mathbf{n})+1}$. If $S \neq U_n$ It is not difficult to see that one can construct a terminal object in the connected component of $\mathbf{RO}(\mathbf{n})^{\mathbf{RO}(\mathbf{n})+1}$ out of terminal object in $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$. Let $t \in \mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$ be such an object. We construct an object t' from $\mathbf{RO}(\mathbf{n})^{\mathbf{RO}(\mathbf{n})+1}$ as follows: for any two K -vertices in t connected by an edge, or for a leave or root attached to a K -vertex we replace this edge or leave by a linear tree with only one vertex. We assign the colour X to this new vertex. We have a morphism from t to t' generated by the unit of the n -operad. For example:



For $S = U_n$ a typical terminal object in the connected component is a linear tree whose vertices are decorated by U_n and whose colours are alternating between X and K , starting with X and ending with X .



Obviously those new objects are terminal in the connected components of $\mathbf{RO}(\mathbf{n})^{\mathbf{RO}(\mathbf{n})+1}$. \square

15.23. Relation to Tamarkin's bad cells in Fulton-Macpherson operad.

The difficulties with the case of n -operads ($n \geq 2$) in comparison with monoids and nonsymmetric operads (and many other type of operads) are closely related to the existence of so called bad cells in Fulton-Macpherson compactification of real configurations spaces discovered by Tamarkin [4, 33]. For example, the object of $\mathbf{NO}(\mathbf{n})_S^{\mathbf{NO}(\mathbf{n})+1}$ ($n = 2$) represented on Figure 7 is terminal in its connected component.

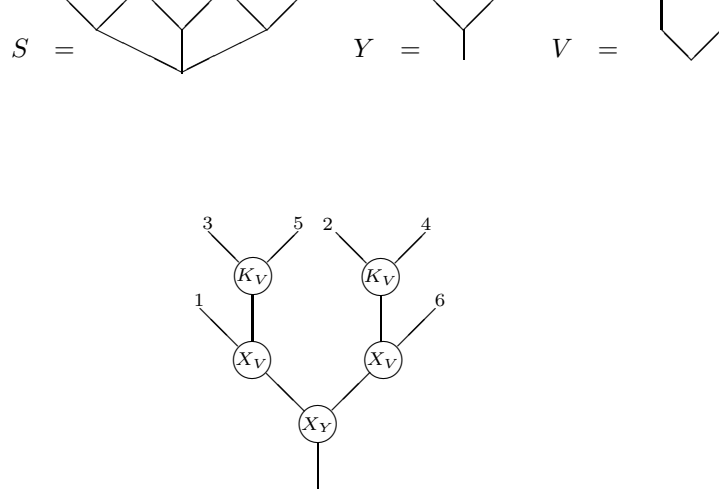


FIGURE 7. Terminal object in a connected component of $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$ which corresponds to a Tamarkin's bad cell.

It was shown in [4] that this figure corresponds exactly to the first bad cell in Fulton-Macpherson operad \mathbf{fm}^2 . One can show that certain cells of maximal dimensions in \mathbf{fm}^n correspond to terminal objects in $\mathbf{NO}(\mathbf{n})^{\mathbf{NO}(\mathbf{n})+1}$ but in general, we don't know an explicit characterisation of such cells and as a consequence an explicit formula for semifree coproducts of n -operads for $n \geq 2$.

15.24. Free n -operad monad for $n \geq 2$ is not tame. Let $O(n)$ be free n -operad monad. It is also polynomial, which was essentially proved in [5] (see also [9]). As we will see the polynomial monad $O(n)$ is not tame for $n \geq 2$ and this creates an obstruction for the existence of model structure in general.

Some connected components of $\mathbf{O}(\mathbf{n})^{0(\mathbf{n})+1}$ do not have terminal objects but do contain weak terminal objects. This is because we have to include trees with stamps (vertices with no leaves) in the description of the free n -operad in this case. An effect is that colimits over $\mathbf{O}(\mathbf{n})^{0(\mathbf{n})+1}$ can be more complicated then in reduced case. Weak equivalences can be unstable with respect to these colimits and this creates an obstacle for the existence of transferable model structure.

More precisely. Let $n = 2$. The category $\mathbf{O}(\mathbf{n})_0^{0(\mathbf{n})+1}$ has as objects all planar coloured decorated trees without leaves. All such trees which contain exactly two K -vertices which are stamps form a full connected subcategory of $\mathbf{O}(\mathbf{n})_0^{0(\mathbf{n})+1}$. It is not hard to see that this subcategory contains a final subcategory which consist of two objects



where Y, V are 2-ordinals as in (15.23). There are exactly two nontrivial morphisms between these objects generated by two possible quasibijections σ_0 and σ_1 from V to Y . The implication is that the 0-component of the semifree coproduct of 2-operads will contain a summand isomorphic to the coequalizer of

$$X_Y \otimes K_0 \otimes K_0 \xrightleftharpoons[d_0]{d_1} X_V \otimes K_0 \otimes K_0,$$

where

$$d = X(\sigma_0) \otimes id_{K_0 \otimes K_0}$$

$$d' = X(\sigma_1) \otimes \tau_{K_0 \otimes K_0},$$

and τ is the symmetry morphism in \mathcal{E} .

Let now \mathcal{E} be the category of chain complexes over a field \mathbf{k} of characteristic two. Let K be a nonreduced 2-collection such that $K_0 = C$, where C is an acyclic cofibrant object in \mathcal{E} and $K_T = 0$ for all $T \neq 0$. Let X be a 2-operad with $X_T = \mathbf{k}$ for all $T \in Ord(2)$. The above calculations show that the coproduct $F(K) \vee X$ will contain a summand $C \otimes C / \mathbb{S}_2$ which is not necessarily an acyclic complex, moreover, it is not hard to find a C such that 0-homology of $C \otimes C / \mathbb{S}_2$ will be $\mathbf{k} \oplus \mathbf{k}$. Hence, $X \rightarrow F(K) \vee X$ can not be a weak equivalence. If the category of 2-operads in \mathcal{E} admits transferred model structure this map must be a weak equivalence as trivial cofibration are closed under pushouts.

Similar obstructions for the existence of model structure on n -operads exist for any $n \geq 2$.

16. HIGHER EXTENSION OF POLYNOMIAL MONADS

With any polynomial monad T one can associate another polynomial monad T^+ the so called Baez-Dolan $+$ -construction [2, 35, 32]. The easy example of this process is free monoid monad, which is the $+$ -construction on the identity monad on Set , and free nonsymmetric operad monad, which is the $+$ -construction of the free monoid monad. In general the main property of the $+$ -construction is that the algebras of T^+ are exactly cartesian monads over T , so, for example, cartesian monads over free monoid monad are nonsymmetric operads.

The following Theorem can be easily proved by the same method as we used for free monoid monad. We skip the details because it would require to reproduce a sizeable piece of material from [32].

Theorem 16.1. *A polynomial monad T is tame if and only if T^+ is tame polynomial monad.*

This theorem allows to develop homotopy theory of coherent algebra of any tame polynomial monad by considering a cofibrant resolution Ass_∞^T of the T^+ -algebra Ass^T which has all components equal the unit of \mathcal{E} . So, if $T = M$ is free monoid monad then $Ass^M = Ass$ the nonsymmetric operad for monoids and Ass_∞^M is

classical Ass_∞ . The operad $Ass_\infty^{M^+}$ is the operad whose algebras are nonsymmetric A_∞ -operads.

APPENDIX A. GRAPHS AND TREES

In this section we give necessary formal definition of graphs, trees, corollas and substitution of a graph to a vertex of another graph.

Definition A.1. A graph G is a polynomial:

$$V \xleftarrow{\pi} \bar{H} \xrightarrow{i} H \xrightarrow{q} E$$

such that

- The sets V, H, \bar{H}, E are finite;
- $q^{-1}(e)$ has exactly two elements for each $e \in E$;
- i is a monomorphism.

The elements from E are called edges, the elements of H are called flags and the elements of V are called vertices of G . We always will think that \bar{H} is just a subset of H , and we will call the elements of $H \setminus \bar{H}$ free flags of G .

An edge $e \in E$ is called trivial if $i^{-1}q^{-1}(e) = \emptyset$ and it is called an external edge if $i^{-1}q^{-1}(e)$ is a one element set. An edge which is not trivial or external is called an internal edge. The boundary of an edge e is the set $\pi(i^{-1}q^{-1}(e))$.

A morphism of graphs is a cartesian morphism of corresponding polynomials.

A linear graph on n vertices $[1, n]$ is a graph which has exactly n -vertices and $n - 1$ internal edges and there exists a linear order on the set of vertices such that the boundary of any edge consists of two consecutive elements. The linear graph $[1, n], n > 1$ has exactly two vertices which have a single flag attached to it. We will call them boundary vertices. A linear tree on n vertices is a graph L_n which has exactly two external edges and such that L_n with these external edges removed is isomorphic to $[1, n]$. The half tree on n -vertices is a graph H_n with exactly one external edge and such that H_n with this external edge removed is isomorphic to $[0, 1]$. A path between two vertices v_1, v_2 of a graph G is a map $[1, n] \rightarrow G$ such that the boundary points go to the vertices v_1, v_2 .

A graph G is *connected* if for any two vertices v there exists a path between them. A graph is called a *tree* if such a path exists and is unique.

Remark A.2. With every graph one can associate a diagram of topological spaces. Consider G as a contravariant functor from the category $\bullet \rightarrow \bullet \leftarrow \bullet \leftarrow \bullet$ to *Set*. Let $el(G)$ be its category of elements. The set of objects of $el(G)$ is $E \sqcup H \sqcup \bar{H} \sqcup V$. The morphisms are generated by the following subset of morphisms: for $e \in E$ there is one morphism to each $h \in q^{-1}(e)$, similarly for each $v \in V$ there is one morphism to each $h \in \pi^{-1}(v)$. Finally, there is exactly one morphism for each $h \in \bar{H}$ from $i(h)$ to h . We have a functor (defined up to isomorphism)

$$Cell(G) : el(G) \rightarrow Top.$$

$$Cell(G)(x) = \begin{cases} [0, 1] & , \quad x \in E, H, \bar{H} ; \\ \{0\} & , \quad x \in V . \end{cases}$$

Let $e \in E$ and $h_1 \neq h_2 \in q^{-1}(e)$. We define the value of $Cell(G)$ on $e \rightarrow h_1$ to be a homeomorphism given by $f(t) = t$ and on $e \rightarrow h_2$ to be a homeomorphism $f(t) = 1 - t$. For a morphism $v \rightarrow h, v \in V, h \in \bar{H}$ the value of $Cell(G)$ is the

inclusion of 0 to $[0, 1]$. Finally, for all morphisms $i(h) \rightarrow h$ the value of $Cell(G)$ is the identity.

The geometric realisation of G is just the colimit of $Cell(G)$. A graph with nonempty set of vertices is connected if and only if its geometric realisation is connected and it is a tree if and only if its geometric realisation is contractible.

Remark A.3. Recall, that in [13, 16, 24] a slightly different definition of a graph has been used: a graph Γ is a map of sets $\pi : H \rightarrow V$ together with an involution $\sigma : H \rightarrow H, \sigma^2 = 1$. We can construct a graph G_Γ

$$V_\Gamma \leftarrow \bar{H}_\Gamma \rightarrow H_\Gamma \rightarrow E_\Gamma$$

in our sense out of Γ as follows. We put $\bar{H}_\Gamma = H, V_\Gamma = V$ and $\pi_\Gamma = \pi$. We take E_Γ equal factor set of H by involution σ . So we have a quotient map $\bar{H}_\Gamma \rightarrow E$. We factorize this map in an injection i_Γ followed by a map q_Γ with the property that each fiber of q_Γ has exactly two elements:

$$\bar{H}_\Gamma \rightarrow H_\Gamma \rightarrow E.$$

The set H_Γ can be constructed as $H \sqcup H^\sigma$ where H^σ is the set of fixed points of σ (so we simply add to H one extra point for each fixed point of σ).

The construction above shows that the category of graphs in the sense of [13, 16, 24] is the full subcategory of our category. The difference is exactly in the treatment of graphs without vertices. Our definition admits a nontrivial graph $0 \leftarrow 0 \rightarrow 2 \rightarrow 1$ (the only connected graph without vertices) whose geometric realisation is an interval $[0, 1]$ whereas in the setting of [13, 16, 24] $V = 0$ forces the graph to be empty whose geometric realisation is an empty set.

Remark A.4. Our notion of graph is equivalent to the notion given by Joyal and Kock in [27], Here a graph is defined as a span in *Set* :

$$E \xleftarrow{s} H \rightarrow V$$

equipped with a fixed point free involution $\sigma : E \rightarrow E$ such that s is an injection. We choose our presentation because it simplifies slightly the construction of geometric realisation and substitution operation.

Definition A.5. A corolla is a graph with one vertex whose all edges are external. That is a graph of the form

$$1 \leftarrow E \xrightarrow{i_1} E \sqcup E \xrightarrow{\nabla} E,$$

where i_1 is the first coprojection and ∇ is the codiagonal.

There is a canonical way to associate a corolla with a graph which we call total contraction. For a graph G its total contraction is the corolla $t(G)$ equal to

$$1 \leftarrow H \setminus \bar{H} \xrightarrow{i_1} (H \setminus \bar{H}) \sqcup (H \setminus \bar{H}) \xrightarrow{\nabla} H \setminus \bar{H}.$$

Each vertex v of a graph G determines a corolla $cor_v(G)$

$$\{v\} \leftarrow \pi^{-1}(v) \rightarrow q^{-1}(q(i(\pi^{-1}(v)))) \rightarrow q(i(\pi^{-1}(v)))$$

which we will call a corolla attached to v . It come equipped with a canonical morphism $cor_v(G) \rightarrow G$.

For a graph G and a vertex $v \in V$ let $G \setminus \{v\}$ be the graph

$$V \setminus \{v\} \leftarrow \bar{H} \setminus \pi^{-1}(v) \rightarrow H \rightarrow E.$$

We have an inclusion

$$\text{cor}_v(G) \setminus \{v\} \rightarrow G \setminus \{v\}.$$

Let G' be another graph $V' \leftarrow \bar{H}' \rightarrow H' \rightarrow E'$ and let ρ be a bijection between the set of free flags of the corolla $\text{cor}_v(G)$ and the set of free flags of the graph G' . Such a bijection produces a unique morphism of graphs

$$\text{cor}_v(G) \setminus \{v\} \rightarrow G'.$$

To see this we observe that a map between the sets free flags induces a unique map between the sets of edges by naturality. Then we have no choice how to define this map on the rest of the flags of the corolla.

Hence the bijection ρ induces a span of maps of graphs:

$$G' \leftarrow \text{cor}_v(G) \setminus \{v\} \rightarrow G \setminus \{v\}.$$

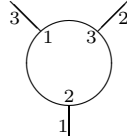
An insertion of G' to $v \in G$ along the bijection ρ is the graph $G' \circ_v^{\rho} G$ obtained as a pushout of this diagram.

A *rooted tree* is a tree with a distinguished external edge called root. Other external edges of a tree are called leaves. In a rooted tree a corolla attached to each vertex can be equipped with rooted structure in a canonical way. The leaves of this corolla will be called the incoming edges of the vertex. The root of the corolla is called the outgoing edge of the vertex.

Let n be the set $\{1, \dots, n\}$ and let $[n]$ be the graph

$$n \xleftarrow{id} n \xrightarrow{i_1} n \sqcup n \xrightarrow{\nabla} n$$

A *partially ordered graph* G is a graph G equipped with a morphism of graphs $[n] \rightarrow G$, which is a bijections on edges. Such a morphism can be understood as an introduction of a linear order on the set of free flags of G . An *ordered graph* G is a partially ordered graph G equipped with partially ordering of all corollas $\text{cor}_v(G)$ for each vertex $v \in G$. Intuitively we can think that such an ordered graph has linear order on the set of external edges as well as linear orders on each set of all edges attached to a vertex. In particular, an ordered corolla G has two different linear order attached to it. One order is on the set of free flags, and the other is on the set of free flags of $\text{cor}_v(G)$.



An isomorphism between two ordered graphs is any isomorphism between underlying graphs which preserves all orderings.

An *ordered rooted tree* is a rooted tree equipped with an ordering in a way that the root is the first element for every linear order. It means, of course, that we can forget about the root of the tree and roots of the attached corollas and order only leaves of the tree and incoming edges for each vertex.

For any tree T and its vertex v there is a well defined function from the free flags of T to the free flags of $\text{cor}_v(T)$. This function is constructed as follows. Given a free flag h of T there is a unique map from a half tree H_n to T which maps the unique flag of H_n to h and the boundary point of H_n to v . In particular, it maps the unique flag attached to the boundary point to a flag attached to v which determines uniquely a free flag of $\text{cor}_v(T)$. If T is ordered this is a map between

linearly ordered set. We will say that an ordered tree T is a *planar tree* if for any vertex v this map preserves linear order up to cyclic permutation, which means that this it becomes an order preserving map after a cyclic permutation of the set of free flags of $\text{cor}_v(T)$.

In rooted tree T there is a analogously defined function from the set of leaves above a vertex $v \in T$ and the set of incoming edges of v . We call T a *planar rooted tree* if this function preserves linear order.

Ordered graphs admit an operation of insertion which depends only of compatibility conditions between graphs. We use linear order on free flags of G' and free flags of the $\text{cor}_v(G)$ to produce a unique bijection which preserves these orders. Hence, we can speak about insertion of G' to the vertex v and we have an unambiguous notation $G' \circ_v G$. We can easily check that the subcategories of ordered trees, rooted trees, planar trees, planar rooted trees, linear trees are closed under this operation and, moreover, the insertion operation respects isomorphisms in all of these categories, hence, we have a well defined substitution operation on isomorphism classes.

An *oriented graph* is a graph with chosen element in each fiber $q^{-1}(e), e \in E$. Such a chosen element can be understood as a choice for an orientation for each edge thinking about chosen element as the beginning of the edge. Morphisms of oriented graphs are required to preserve chosen elements. We also have *oriented ordered graphs* which are oriented graphs with ordering. For an oriented graph G and a vertex v one define orientation on $\text{cor}_v(G)$ in an obvious way. One can now define insertion operation as in graphs but asking the bijection ρ to be orientation preserving. Any rooted tree admits two canonical orientation from top to bottom or vice versa. We always will think that rooted tree is oriented from top to bottom. We also have oriented version of linear graph $[1, n]$ with the orientation going from p to $p-1$. A loop in a oriented graph G is any map of oriented graphs $s : [1, n] \rightarrow G$ for which $s(1) = s(n)$.

A *loop free graph* is a graph which does not admit a loop. Loop free graphs are closed under operation of insertion.

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